Supplement: Solving the obstacle problem

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Contents

We define the continuous version of the Diaconis-Fulton smash sum, as described in Levine and Peres [4], and prove that the definition makes sense and satisfies the axioms in the main paper. The proof follows Sakai [6].

1 The axioms

The axioms in the main paper are:

Translation invariance. For any $c \in \mathbb{R}^d$, $(A+c) \oplus (B+c) = (A \oplus B) + c$.

Rotation invariance. For any orthonormal matrix $U : \mathbb{R}^d \to \mathbb{R}^d$,

$$UA \oplus UB = U(A \oplus B).$$

Commutativity. $A \oplus B = B \oplus A$.

Associativity when possible. If both sums $A \oplus B$, $B \oplus C$ are bounded, then: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Monotonicity. $A \oplus B \supseteq A, B$. If $A \subseteq C$, then $A \oplus B \subseteq C \oplus B$.

Conservation of mass. Let λ be Lebesgue measure. Then the λ-measure of $A \oplus B$ is equal to $\lambda(A) + \lambda(B)$.

2 Definitions

2.1 Subharmonicity

Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $C \subseteq \Omega$ be an open subset.

Recall that a function $h : \Omega \to \mathbb{R} \cup \{+\infty\}$ is *subharmonic on C* if it is locally integrable on C and:

(a) it is upper semicontinuous, i.e. $\limsup_{y \to x} h(y) \le h(x)$, and

(b) for every point $x \in C$ and sufficiently small $r > 0$, the value of h at x is less than or equal to the average of h on $B_r(x)$.

We can write this on one line as

$$\limsup_{y \to x} h(y) \le h(x) \le \inf_{r>0} \frac{1}{\lambda(B_r)} \int_{B_r(x)} h(y)\,dy,$$

and the left side is at least as big as the right, so those are equalities.

Exercise. If h and $-h$ are subharmonic on an open set, then h is twice continuously differentiable on that set and $\nabla^2 h = 0$.

2.2 Quadrature sets

We'll say that a function w is a *weight function* if it is bounded, nonnegative, and measurable. It is a *properly supported* weight function if $w \geq 1$ on some bounded open set and $w = 0$ outside that set.

A *quadrature set* for a properly supported weight function w is a bounded open set Q with the property that $\int hw\,dx \leq \int_Q h\,dx$ for every integrable subharmonic function h on Q.

In particular, $\int w\,dx = \lambda(Q)$, because the two functions ± 1 are subharmonic and integrable, and the two inequalities give us an equality. The centre of mass is the same, $\int x_i w\,dx = \int_Q x_i\,dx$, because all the functions $\pm x_i$ are subharmonic and integrable. The moment of inertia of Q is no less than that of w: $\int |x|^2\,w\,dx \leq \int_Q |x|^2\,dx$, because $|x|^2$ is subharmonic.

This definition is opaque; here is some intuition about what it means.

2.2.1 Intuition about quadrature sets

Here is a problem in the theory of Brownian motion that gives some intuition. Suppose I have a properly supported weight w. Imagine that I place particles in $\mathbb{R}^d$ with density measure $w\,d\lambda$, and then run a stopped Brownian motion on each particle.

If the particle that starts at x is stopped at T_x, then the final distribution of particles has density measure μ with $\mu(A) = \int \mathbb{P}_x[B_x(T_x) \in A]\,w(x)\,dx$.

Let Q be a bounded open set. Let $B_x(t)$ be Brownian motion started at x. Pick a family of stopping times T_x for $x \in Q$ so that no time is greater than the first exit time $\inf\{t : B_x(t) \notin Q\}$. Then

$$\mathbb{E}_x h(B_x(T_x)) \geq h(x)$$

for any bounded subharmonic h on Q. Integrating over the weight function gives the family of integral inequalities

$$\int \mathbb{E}_x h(B_x(T_x))\,w(x)\,dx \geq \int h(x)w(x)\,dx.$$

Suppose there is a choice of stopping times so that the final density of particles is $\mathbb{1}_Q$, meaning that for every measurable set $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\int \mathbb{P}[B_x(T_x) \in A]\,w(x)\,dx = \lambda(A \cap Q).$$

In that case, we have the quadrature set property for Q:

$$\int_Q h(x)\,dx = \int \mathbb{E}_x h(B_x(T_x))\,w(x)\,dx \geq \int h(x)w(x)\,dx.$$

[not finished]

2.3 Bulky open sets

Recall that two open sets A, B are *essentially equal* if $\lambda(A \,\Delta\, B) = 0$, and that an open set is *bulky* if it contains every open set essentially equal to it.

If A is an open set, there is exactly one bulky set that is essentially equal to it, and we denote that set by $[A] = \bigcup\{B \mid B \text{ ball}, B \subseteq A\}$.

2.4 Definition of the sum

The definition of the smash sum relies on the existence and uniqueness theorems for quadrature sets of properly supported weight functions.

First, quadrature sets are unique up to sets of measure zero.

Corollary 4. *Quadrature sets are essentially unique. If C and D are two quadrature sets for the same weight function, then $\lambda(C \,\Delta\, D) = 0$.*

Second, we will say a weight function is *properly supported* if there is a bounded open set Ω with the property that w is at least 1 on Ω and identically zero outside it. The second theorem says that a bounded, properly supported weight function has at least one quadrature set.

Corollary 33.
If w is bounded and properly supported, there is a quadrature set for it.

Definition of the sum. If A and B are bounded open sets, let the sum of A and B be the unique bulky quadrature set for the weight $w = \mathbb{1}_A + \mathbb{1}_B$.

That weight is properly supported on $A \cup B$, so there is a quadrature set for it by Corollary 33 quoted above.

3 Green's function

We define Green's function for the Laplacian in terms of Brownian motion as on page 80, section 3.3 of Mörters and Peres [4]. It is a general definition which works on any bounded open set. The details are sketched here.

Let B_y be Brownian motion started at y. The probability density function of $B_y(t)$ is $\mathbb{P}[B_y(t) \in A] = \int_A \mathfrak{p}(t; x, y)\, dx$, where

$$\mathfrak{p}(t; x, y) = (2\pi t)^{-d/2} \exp\left(-||x - y||^2/2t\right).$$

Here $||x - y||$ is Euclidean distance. This function is symmetric in x and y, and it is called the unrestricted transition kernel.

It's a solution of the heat equation,

$$\left(\partial_t - \frac{1}{2}\nabla_x^2\right)\mathfrak{p}(t; x, y) = \left(\partial_t - \frac{1}{2}\nabla_y^2\right)\mathfrak{p}(t; x, y) = 0.$$

Notice the factor of 1/2. We want our definition to be consistent with the usual mathematical one, where the constant is 1, so we'll speed up the time of the Brownian motion by a factor of two.

Define the *free-space Green's function* for $d \geq 3$ as:

$$G^{(d)}(x, y) := \int_0^\infty \mathfrak{p}(2t; x, y)\, dt = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} ||x-y||^{2-d}.$$

In two dimensions, define it as the parametric limit

$$\lim_{d \to 2} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \Big[||x-y||^{2-d} - 1 \Big] = -\frac{1}{2\pi} \log ||x-y||.$$

It's symmetric, $G(x,y) = G(y,x)$, and it's smooth and harmonic in both variables on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$, meaning $\nabla_x^2 G(x,y) = \nabla_y^2 G(x,y) = 0$.

Lemma 1. *In the distributional sense,* $\nabla^2 G(x, \cdot) = -\delta_x$.

Sketch of proof. Let h be a compactly supported test function, and integrate $G\nabla^2 h = \nabla \cdot (G\nabla h - h\nabla G)$ over $\mathbb{R}^d \setminus B_\varepsilon(x)$.

Use Stokes's theorem to turn it into an integral on $\partial B_\varepsilon(x)$, estimate that using $\frac{\partial G}{\partial r} = -1/(C_d r^{d-1})$ and the continuity of h, and then take $\varepsilon \to 0$. $\square$

The *restricted transition kernel*, the transition kernel of a Brownian motion killed when it leaves an open set C, is the function $\mathfrak{p}_C$ satisfying

$$\mathbb{P}\Big[B_y(t) \in A \text{ and } B_y([0,t]) \subseteq C\Big] = \int_A \mathfrak{p}_C(t; x, y)\, dx.$$

Such a function exists, and it is symmetric in the two space variables; this is proven in the above section of Mörters and Peres.

Set $G_C(x,y) = \int_0^\infty \mathfrak{p}_C(t; x, y)\, dt$. This is our function, the *Green's function of the open set* C. From the properties of $\mathfrak{p}_C$ and Brownian motion, we get:

- $G_C \geq 0$.
- $G_C(x, y)$ when x and y are in the same component of C.
- $G_C(x, y) < \infty$ if $x \neq y$, and $G_C(x, x) = +\infty$.
- $G_C(x, y) = G_C(y, x)$.
- $G_C(\cdot, y)$ is harmonic on $C \setminus \{y\}$, and superharmonic on C.

It turns out also that $G(x, y) - G_C(x, y)$ is harmonic on $C \times C$.

3.1 The extended Green's function

Writers who describe G_D as "the Green's function" should be condemned to differentiate the Lebesgue's measure using the Radon-Nikodym's theorem.

— Joseph Doob

Even so, there is a subharmonic extension of Green's function to $\mathbb{R}^d \setminus \{y\}$, which we refer to as *the extended Green's function.*

Theorem 2. *If C is a bounded open set with Green's function G_C, there is an extension $g_C : \mathbb{R}^d \times C \to \mathbb{R} \cup \{\infty\}$ so that:*

- $g_C(x, y) = G_C(x, y)$ *when* $x \in C$. *(So it is superharmonic on* C*.)*
- $g_C(\cdot, y)$ *is subharmonic on* $\mathbb{R}^d \setminus \{y\}$.
- $g_C(\cdot, y)$ *is zero almost everywhere on* C^c.

Proof. We just give a reference. This is (c) of Doob's Theorem 1.VII.4 [1]. (Our set is bounded, so it is Greenian and the theorem applies. A polar set has measure zero, so when something is true "quasi everywhere," in other words on the complement of a polar set, it holds almost everywhere a fortiori.)

4 Monotonicity of quadrature sets

In the last section we saw that, for any bounded open set C and $y \in C$, there is a function $g_C(\cdot, y)$ which is nonnegative everywhere, positive on the component of y in C, zero almost everywhere outside C, and superharmonic on C and subharmonic on $\mathbb{R}^d \setminus \{y\}$.

The existence of that function implies that quadrature sets get larger as their weight functions get larger:

Theorem 3. *If* $w \le w'$ *are two nonnegative measurable functions and* C *is a quadrature set for* w *and* D *is a quadrature set for* w'*, then* $\lambda(C \setminus D) = 0$.

Proof. Recall that quadrature sets are by definition open. Let E be a component of C that is not contained in D; if there is no such component, we are done. Fix $y \in E \setminus D$. Let $h_C(x) := g_C(x, y)$. It is nonnegative everywhere. By the definition of a quadrature set,

$$\int_C h_C \, dx \le \int h_C w \, dx \le \int h_C w' \, dx \le \int_D h_C \, dx.$$

So $\int_{C \setminus D} h_C \, dx \le \int_{D \setminus C} h_C \, dx$, but that second integral is zero because $h_C \equiv 0$ almost everywhere on C^c. Therefore $\int_{C \setminus D} h_C \, dx$ is zero too. Green's function is strictly positive on E, so $\lambda(E \cap (C \setminus D))$ must be zero.

An open set has only countably many components. Take the union over all components E of C to get the result.

Corollary 4. *Quadrature sets are essentially unique.*

Proof. If C, D are quadrature sets for w, then $\lambda(C \setminus D) = \lambda(D \setminus C) = 0$.

5 Positivity of the Laplacian

5.1 Distributions and notation

Let Ω be an open subset of $\mathbb{R}^d$.

A *distribution* on Ω is a continuous linear map $C_c^\infty(\Omega) \to \mathbb{R}$, where $C_c^\infty(\Omega)$ has the usual topology of uniform convergence of all derivatives on compact sets. Let $D'(\Omega)$ be the vector space of distributions. Recall some standard facts. If $\varphi \in D'(\Omega)$, then the derivative $\partial_i \varphi$ is the distribution that maps h to $-\varphi(\partial_i h)$, the minus sign being the result of "partial integration." The Laplacian of a distribution φ is $\nabla^2 \varphi : h \mapsto \varphi(\nabla^2 h)$.

Notation.

If μ is a locally finite measure on Ω, or a signed measure, let $d\mu$ be the distribution $h \mapsto \int h(x)\,\mu(dx)$. This integral is well-defined since h is supported on a compact subset of Ω and μ is finite on that subset by compactness.

If f is a locally integrable function on Ω, let $f\,d\lambda$ be $h \mapsto \int h(x) f(x)\,dx$, where as before λ is Lebesgue measure.

A word about this notation. It's more common to think of measures and (equivalence classes of) locally integrable functions as being contained in the space of distributions, so that a measure or function would be written simply as μ or f. But we will want to compare and add the two types of distributions. To avoid anti-intuitive expressions like "$\mu \le 1$" or "$\nu \le 1 - w$" in which one term looks like a signed measure, the other term looks like a function, and the proofs look as if we are about to fail undergraduate measure theory, we write our distributions to make it clear that they are in a common vector space.

5.2 Subharmonic functions on an open set

A distribution $\psi \in D(\Omega)$ is said to be *positive* if $\psi[h] \ge 0$ for every nonnegative test function $h \in C_c^\infty(\Omega)$, and *negative* if $-\psi$ is positive.

If f is a twice continuously differentiable function which is subharmonic on an arbitrary subset of Ω, then $\nabla^2 f \ge 0$ at every point in that set. Our goal in this section is to prove a similar result for general functions.

Theorem 5. *If f is subharmonic on Ω, then $\nabla^2(f\,d\lambda)$ is positive on Ω.*

Proof. Fix $h \in C_c^\infty(\Omega)$ with $h \ge 0$. Let $\varphi \ge 0$ be a smooth function on $\mathbb{R}^d$ that is zero outside the unit ball and has unit integral, and $\varphi = n^{-d}\varphi(x/n)$. Then $f * \varphi_n$ is defined on the support of h for large enough n.

The functions $f * \varphi_n$ are smooth, so derivatives are defined in the classical sense, and they are subharmonic, because

$$
\begin{aligned}
f * \varphi_n(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_n(y)\,dy &\le \int \left[\frac{1}{\lambda(B_r)} \int_{B_r} f(x-y+z)\,dz\right] \varphi_n(y)\,dy \\
&= \frac{1}{\lambda(B_r)} \int_{B_r} f * \varphi_n(x+z)\,dz.
\end{aligned}
$$

So $\nabla^2(f * \varphi_n)$ is defined in the ordinary sense and is nonnegative.

Finally, $f * \varphi_n \to f$ in L^1 on $\operatorname{supp} h$. Therefore $\int (f * \varphi_n) h \to \int f h$, so

$$
\begin{aligned}
\nabla^2(f\,d\lambda)(h) = \int f\,\nabla^2 h\,dx &= \lim_{n\to\infty} \int (f * \varphi_n)\,\nabla^2 h\,dx \\
&= \lim_{n\to\infty} \int \nabla^2(f * \varphi_n)\,h\,dx \\
&\ge 0.
\end{aligned}
$$

That is true for every nonnegative $h \in C_c^\infty(\Omega)$, so $\nabla^2 f$ is positive on Ω. $\square$

Lemma 6. *If ψ is a positive distribution on Ω, then there is a locally finite measure μ on Ω with $\psi(h) = \int h\,d\mu$ for every test function $h \in C_c^\infty(\Omega)$.*

Proof. See Rudin [5], chapter 6 exercise 4. We sketch the proof.

Let ψ be a positive distribution. If $K \subset\subset \Omega$ is a compact subset, then there is a nonnegative $h_1 \in C_c^\infty(\Omega)$ that is identically 1 on K. Positivity says $0 \le \psi(h) \le \psi(h_1)$ if $h \in C_c^\infty(K)$ and $0 \le h \le 1$. An easy approximation argument tells us that $|\psi(h)| \le \psi(h_1)\|h\|_{C(K)}$. By the Riesz representation theorem, there is a finite measure μ_K with $\psi(h) = \int h\,d\mu_K$ for $h \in C_c^\infty(K)$.

Take a net of compact sets increasing to Ω: the measures are consistent on those sets, and we get a limit measure μ with $\psi(h) = \int h\,d\mu\ \forall h \in C_c^\infty(\Omega)$. $\square$

Corollary 7. *If f is subharmonic on Ω, then there exists a locally finite measure μ on Ω with $\nabla^2(f\,d\lambda) = d\mu$.*

Proof. If f is subharmonic on Ω, then $\nabla^2 f$ is positive on Ω by Theorem 5, so it is a locally finite measure on that set. $\square$

We now study the relationship between averages on concentric balls and the distributional Laplacian, and use that to handle a function which is subharmonic on an arbitrary measurable set.

5.3 The spherical average function

Suppose f is locally integrable, x is a point in Ω, and $0 < r < R := d(x, \Omega^c)$. Let the average on the sphere of radius r around x be denoted by

$$L_f(x;r) := \frac{1}{C_d} \int_{|z|=1} f(x+rz)\,dz.$$

Here $C_d = \int_{|z|=1} 1\,dz = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere in $\mathbb{R}^d$. This function isn't defined for every r, but the lemma below tells us that it's defined for almost every $r \in (0, R)$ and it's locally integrable on $(0, R)$.

Lemma 8. *Let Ω be an open set, $x \in \Omega$, $R := d(x, \Omega^c)$.*

If f is locally integrable, then $L_f(x;r)$ is defined for almost every radius $r \in (0, R)$, and $\int_0^s r^{d-1}|L_f(x;r)|\,dr < \infty$ for $0 < s < R$.

Also, $L_f(x;r)$ is locally integrable on $(0, R)$.

Proof. Let $0 < s < R$. $\overline{B_s(x)}$ is compact, so $\int_{B_s(x)} |f|\,dx < \infty$.

Write this as a double integral:

$$\infty > \int_{B_s(x)} \left|f(y)\right| dy = \int_0^s \left[\int_{|z|=1} \left|f(x+rz)\right| r^{d-1}\,dz \right] dr.$$

Tonelli's theorem tells us that the integral in brackets is finite for almost every radius on $(0, s)$. Take $s_n \nearrow R$ to see that the integral is finite for almost every $r \in (0,R) = \bigcup_n (0, s_n)$. Therefore, $L_f(x;r) := \frac{1}{C_d} \int_{|z|=1} f(x+rz)\,dz$ is well-defined for almost every r in $(0,R)$, and Jensen's inequality says that

$$\begin{aligned}\int_0^s r^{d-1}|L_f(x;r)|\,dr &= \frac{1}{C_d}\int_0^s r^{d-1}\left|\int_{|z|=1} f(x+rz)\right|\,dz\,dr \\ &\leq \frac{1}{C_d}\int_0^s \int_{|z|=1} |f(x+rz)|\,r^{d-1}\,dz\,dr \\ &< \infty.\end{aligned}$$

We must also prove local integrability. Let $r \in (0,R)$, and choose real numbers t, s with $0 < t < r < s < R$. Then $(r/t)^{d-1} \geq 1$ when $r \in (t,s)$, so

$$\int_t^s |L_f(x;r)|\,dr \leq \int_t^s \left(\frac{r}{t}\right)^{d-1} |L_f(x;r)|\,dr < \infty.$$

That's true for any $r \in (0,R)$, so $L_f(x;r)$ is locally integrable. □

The function $L_f(x;\rho)$ is related to the distributional Laplacian by an integral equality. We explore that in the next section.

5.4 Spherical averages and the distributional Laplacian

Let f be a locally integrable function on Ω. The lemma below tells us that we can find certain integrals over the spherical average function $L_f(x;r)$ by evaluating the distributional Laplacian $\nabla^2(f\,d\lambda)$ at a certain function.

Lemma 9. *Let f be locally integrable on Ω and $x \in \Omega$. Let $R > 0$.*

If $\eta \in C_c^\infty(0,R)$ is nonnegative, there is a nonnegative test function $h \in C_c^\infty(\Omega)$ with $\nabla^2(f\,d\lambda)(h) = -\int_0^R \eta'(r) L_f(x;r)\,dr$.

Note. If $\eta \in C_c^\infty(0,R)$, $\eta \geq 0$, then $\eta'(r)$ is a signed weight function that's smooth and compactly supported on $(0,R)$, and has total weight zero with the positive part of the weight "left of" the negative part in the sense that $\int_0^s \eta'(r)\,dr$ is nonnegative for $0 < s < r$. The converse is also true: any function like that will give us a suitable $\eta \in C_c^\infty(0,R)$, $\eta \geq 0$ by integration.

Proof. We extend η to $\mathbb{R}_{>0}$ by setting it to zero when $r \geq R$.

Set the test function to $h(y) := H(|y-x|)$, where

$$H(r) := \int_r^\infty \frac{\eta(\rho)}{C_d \rho^{d-1}}\,d\rho.$$

Then $H(r)$ is smooth, constant near 0, and zero in a neighbourhood of $[R,\infty)$, so h is smooth and supported on a compact subset of $B_R(x) \subseteq \Omega$. That means it is a test function for distributions on Ω, and the expression $\nabla^2(f\,d\lambda)(h)$ makes sense.

We prove $\nabla^2(f\,d\lambda)(h) = -\int_0^R \eta'(r) L_f(x;r)\,dr$. First, the test function is radially symmetric around x, which means that its Laplacian is

$$\nabla^2 h = \frac{1}{r^{d-1}}\frac{d}{dr}\left[r^{d-1}\frac{d}{dr}H(r)\right]$$

where $r = |y - x|$. From the formula above, $r^{d-1} H'(r) = -\eta(r)/C_d$, so Plugging this in with $r = |y - x|$,

$$\begin{aligned}
\nabla^2(f\,d\lambda)(h) &= \int f(y)\,\nabla^2 h(y)\,dy \\
&= \int f(y)\,\frac{1}{r^{d-1}}\frac{d}{dr}\left[r^{d-1}\frac{d}{dr}H(r)\right]dy \\
&= -\int f(y)\,\frac{1}{r^{d-1}}\frac{\eta'(r)}{C_d}\,dy.
\end{aligned}$$

Write $y = x + rz$ where $r > 0$ and $|z| = 1$.

$$\begin{aligned}
\nabla^2(f\,d\lambda)(h) &= -\int_0^\infty \int_{|z|=1} f(x+rz)\,\frac{1}{r^{d-1}}\frac{\eta'(r)}{C_d}\,r^{d-1}\,dz\,dr \\
&= -\int_0^\infty \int_{|z|=1} f(x+rz)\,\frac{\eta'(r)}{C_d}\,dz\,dr \\
&= -\int_0^\infty \eta'(r) L_f(x;r)\,dr.
\end{aligned}$$

That's what we are trying to prove. □

The key features of this lemma are that h is nonnegative, and that we have an explicit formula for it. In the rest of this section we'll see how to choose η to get some useful information out.

5.5 The difference of averages on two concentric balls: choosing functions for approximation

Let $A_f(x;r)$ be the average of f on $B_r(x)$,

$$A_f(x;r) := \frac{1}{\lambda(B_r)} \int_{B_r(x)} f(y)\,dy.$$

Let $0 < t < s$. We construct explicit functions η_m to use in Lemma 9 with

$$A_f(r;t) - A_f(r;s) = \lim_{m\to\infty} \int_0^\infty \eta_m'(r) L_f(x;r)\,dr \tag{1}$$

It makes intuitive sense that such functions should exist. We know from the note for Lemma 9 that we can use any smooth signed weight function supported in $(0, R)$ with total weight zero where the positive part of the weight is left of the negative weight, by integrating it. But

$$\begin{aligned}
A_f(r;t) &= \frac{1}{\lambda(B_t)} \int_{B_t(x)} f(y)\,dy \\
&= \int_0^t \int_{|z|=1} f(x+rz)\,r^{d-1}\,dz\,dr \Bigg/ \int_0^t \int_{|z|=1} r^{d-1}\,dz\,dr \\
&= \int_0^t \frac{dr^{d-1}}{t^d} L_f(x;r)\,dr,
\end{aligned}$$

so we can write the difference of averages as

$$A_f(r;t) - A_f(r;s) = \int_0^\infty \left[\mathbb{1}_{r<t} \frac{dr^{d-1}}{t^d} - \mathbb{1}_{r<s} \frac{dr^{d-1}}{s^d} \right] L_f(x;r)\, dr.$$

Let $\chi(r) := \min\{1, r^d/t^d\} - \min\{1, r^d/s^d\}$, the integral of the expression in brackets. Then χ is nonnegative and supported on $[0, s]$, so we can hope to use it in Lemma 9. It isn't smooth, and its support goes all the way to zero, but it should be possible to get around that with approximation.

The main obstacle is that $L_f(x;r)$ might have many singularities. All we know about it is that $\int_0^s r^{d-1} |L_f(x;r)|\, dr < \infty$:

$$\begin{aligned}
\int_0^s r^{d-1} |L_f(x;r)|\, dr &= \frac{1}{C_d} \int_0^s r^{d-1} \left| \int_{|z|=1} f(x+rz)\, dz \right| dr \\
&\le \frac{1}{C_d} \int_{B_s(x)} |f(y)|\, dy \\
&< \infty.
\end{aligned}$$

If we aren't careful choosing approximations, we may not be able to prove that the integrals converge. The following lemma picks approximations carefully to avoid trouble with $L_f(x;r)$.

Lemma 10. *Given $0 < t < s$, there are nonnegative functions $\eta_m \in C_c^\infty(0,\infty)$ that converge uniformly to $\chi(r)$, satisfy (1), and that satisfy bounds $\eta_m(r) \le \mathbb{1}_{r<s} \min\{1, r^d/t^d\}$ and $|\eta_m'(r)| \le \mathbb{1}_{r<s} dr^{d-1}/t^d$.*

Let $w(r) = \mathbb{1}_{r<1} dr^{d-1}$. Let $W(r)$ be the indefinite integral $\int_0^r w(\rho)\, d\rho = \min\{1, r^d\}$. Then $\chi(r) = W(r/t) - W(r/s)$. This difference is nonnegative, because $r/t \ge r/s$ and W is increasing, and it's zero for $r \ge s$.

Let $w_m \in C_c^\infty(0,\infty)$ be nonnegative smooth functions with supports contained in $(0,1)$ that increase pointwise to w. Set $W_m(r) := \int_0^r w_m(\rho)\, d\rho$.

Finally, let $\eta_m(r) := W_m(r/t) - W_m(r/s)$. Again, $\eta_m(r)$ is nonnegative because W_m is increasing and constant for $r \ge 1$. The first bound holds:

$$0 \le \eta_m(r) \le W_m(r/t) \le W(r/t) = \min\left\{1, \frac{r^d}{t^d}\right\}.$$

The bound on the derivative holds:

$$|\eta_m'(r)| = \left| \frac{w_m(r/t)}{t} - \frac{w_m(r/s)}{s} \right| \le \max\left\{ \frac{w(r/t)}{t}, \frac{w(r/s)}{s} \right\} \le \frac{dr^{d-1}}{t^d}.$$

It still remains to prove that $\eta_m \to \chi$ uniformly and that (1) holds.

We can get a uniform bound on the difference

$$\begin{aligned}
|\chi(r) - \eta_m(r)| &\le |W(r/t) - W_m(r/t)| + |W(r/s) - W_m(r/s)| \\
&\le 2 \left(\int_0^R w(\rho)\, d\rho - \int_0^R w_m(\rho)\, d\rho \right).
\end{aligned}$$

The second integral converges to the first one by the monotone convergence theorem, so $\max_r |\chi(r) - \eta_m(r)| \to 0$ and $\eta_m \to \chi$ uniformly.

For (1), write the average of f on $B_t(x)$ in terms of L_f:

$$\begin{aligned} A_f(x;t) &= \int_0^t \frac{dr^{d-1}}{t^d} L_f(x;r)\,dr \\ &= \int_0^t \lim_m \frac{w_m(r/t)}{t} L_f(x;r)\,dr. \end{aligned}$$

The functions $(w_m(r/t)/t)L_f(x;r)$ are dominated by $\frac{d}{t^d} r^{d-1}|L_f(x;r)|$, which is integrable on $(0,t)$ by earlier remarks. Use the dominated convergence theorem to move the limit outside the integral:

$$A_f(x;t) = \lim_m \int_0^\infty \frac{w_m(r/t)}{t} L_f(x;r)\,dr.$$

Replace t by s and subtract to get (1).

$$\begin{aligned} A_f(x;t) - A_f(x;s) &= \lim_m \int_0^\infty \left[\frac{w_m(r/t)}{t} - \frac{w_m(r/s)}{s}\right] L_f(x;r)\,dr \\ &= \lim_m \int_0^\infty \eta_m'(r) L_f(x;r)\,dr. \end{aligned}$$

□

5.6 The difference of averages on two concentric balls: a formula for signed measures

Lemma 11. *Suppose f is locally integrable on Ω and $\nabla^2(f\,dx) = d\nu$ where ν is a signed measure. Let $x \in \Omega$ and $0 < t < s < R = d(x, \Omega^c)$.*

Let $\chi(r) := \min\{1, r^d/t^d\} - \min\{1, r^d/s^d\}$ as above, and let

$$h(y) := \int_{|y-x|}^\infty \frac{\chi(\rho)}{C_d \rho^{d-1}}\,d\rho.$$

Then $\int h\,d\nu = A_f(x;s) - A_f(x;t)$.

Proof. Let η_m be the smooth approximations from Lemma 10. Let h_m be the function provided by Lemma 9 with $\nabla^2(f\,dx)(h_m) = -\int_0^\infty \eta_m'(r) L_f(x;r)\,dr$. Taking limits on both sides of that identity,

$$\begin{aligned} \lim_m \nabla^2(f\,d\lambda)(h_m) &= -\lim_m \int_0^\infty \eta_m'(r) L_f(x;r)\,dr \\ &= A_f(x;s) - A_f(x;t). \end{aligned}$$

The left side of this equation is $\lim_m \int h_m\,d\nu$, so all we need to do is prove that $\lim_m \int h_m\,d\nu = \int h\,d\nu$.

Both h_m and h are defined by integrals. Write $h(y) - h_m(y)$ as an integral, take the absolute value, move it under the integral sign, and extend the range of integration to $(0, \infty)$ to get

$$|h(y) - h_m(y)| \leq \left| \int_{|y-x|}^{\infty} \frac{\chi(\rho) - \eta_m(\rho)}{C_d \rho^{d-1}} \, d\rho \right| \leq \int_0^{\infty} \frac{|\chi(\rho) - \eta_m(\rho)|}{C_d \rho^{d-1}} \, d\rho. \quad (2)$$

This is a uniform bound. The bounds on the functions η_m from Lemma 10, and the obvious bound $0 \leq \chi(r) \leq \mathbb{1}_{r<s} r^d / t^d$, tell us that

$$\frac{|\chi(\rho) - \eta_m(\rho)|}{C_d \rho^{d-1}} \leq \mathbb{1}_{\rho < s} \frac{\rho}{t^d}$$

say that the integrand is dominated by $\mathbb{1}_{|\rho|<s} \rho / t^d$, which is certainly integrable. Lemma 10 also says that $\eta_m(\rho) \to \chi(\rho)$ pointwise, so the integrand converges to zero pointwise. Therefore, by the dominated convergence theorem, the right side of equation (2) converges to zero, and $h_m \to h$ uniformly.

This is strong enough convergence to make $\int h \, d\nu = \lim_m \int h_m \, d\nu$ no matter what signed measure we have, so $\int h \, d\nu = A_f(x; s) - A_f(x; t)$. □

This lemma lets us find the difference of averages on concentric balls when we know the Laplacian. Even better, it lets us *estimate* the difference from weak estimates on the Laplacian. To do that, we need to know $\int h \, dx$.

Lemma 12. *Let $0 < t < s$ and $x \in \Omega$. With h defined as above,*

$$\int h(y) \, dy = \frac{1}{2(d+2)} (s^2 - t^2).$$

Proof. We know what the function is, so the proof is straightforward. First,

$$\begin{aligned}
\int_{\mathbb{R}^d} h(y) \, dy &= \int_{\mathbb{R}^d} \left[\int_{|x-y|}^{s} \frac{\chi(\rho)}{C_d \rho^{d-1}} \, d\rho \right] dy \\
&= \int_0^{\infty} \left[\int_r^s \frac{\chi(\rho)}{C_d \rho^{d-1}} \, d\rho \right] C_d r^{d-1} \, dr \\
&= \int_0^s \frac{\chi(\rho)}{\rho^{d-1}} \left[\int_0^{\rho} r^{d-1} \, dr \right] d\rho \\
&= \int_0^s \chi(\rho) \frac{\rho}{d} \, d\rho.
\end{aligned}$$

We have $\chi(\rho) = \max\{1, \rho^d / t^d\} - \max\{1, \rho^d / s^d\}$, so

$$\begin{aligned}
\int_0^s \chi(\rho) \frac{\rho}{d} \, d\rho &= \left[\int_0^t \frac{\rho^{d+1}}{d t^d} \, d\rho + \int_t^s \frac{\rho}{d} \, d\rho \right] - \int_0^s \frac{\rho^{d+1}}{d s^d} \, d\rho \\
&= \frac{t^2}{d(d+2)} + \frac{s^2 - t^2}{2d} - \frac{s^2}{d(d+2)} \\
&= \frac{1}{2(d+2)} (s^2 - t^2).
\end{aligned}$$

□

Remark. We can now get strong bounds on $A_f(x;s) - A_f(x;t)$ from a very small amount of information about $\nabla^2(f\,d\lambda)$.

Suppose $\nabla^2(f\,d\lambda)$ is a signed measure $d\nu$ where $|\nu| \leq C\lambda$. Then

$$|A_f(x;s) - A_f(x;t)| \leq \left|\int h\,d\nu\right| \leq C\int h\,d\lambda = \frac{C}{2(d+2)}(s^2 - t^2),$$

so $\lim_{t\to 0} A_f(x;t)$ exists for every $x \in \Omega$. Let the limit be $\bar{f}(x)$. Take $t \to 0$ above to get the inequality $|A_f(x;s) - \bar{f}(x)| \leq Cs^2/2(d+2)$, which is not only a uniform bound, it's *quadratic* in the radius s, just as strong as if we were in one dimension and knew that $f \in C^2$ and $|f''| \leq C$.[1]

5.7 Subharmonic on average

A function is a *limit of radial averages at* x if it is integrable in a neighbourhood of x and the limit $\lim_{r\to 0} A_f(x;r)$ exists and is equal to $f(x)$. This is strictly weaker than continuity at a point.

It's also weaker than subharmonicity:

Lemma 13. *If a function is subharmonic at a point, it is a limit of radial averages at that point.*

Proof. Let f be subharmonic at x. Then $f(x) \leq A_f(x;r)$ for $r < d(x, \Omega^c)$, so $f(x) \leq \liminf_{r\to 0} A_f(x;r)$, but f is upper semicontinuous at x, so

$$f(x) = \lim_{r\to 0} \max_{B_r(x)} f(y) \geq \limsup_{r\to 0} A_f(x;r).$$

Therefore the lim inf and lim sup are equal, and $f(x) = \lim_{r\to 0} A_f(x;r)$. □

We say that a function is *subharmonic on average* at x if it is a limit of radial averages at x and satisfies condition (b) in the definition of subharmonicity. That is, there exists some small $\varepsilon > 0$ so that

$$\lim_{r\to 0} A_f(x;r) = h(x) \leq \inf_{0<r<\varepsilon} A_f(x;r).$$

Here is an example that shows that this definition is strictly weaker than subharmonicity. Let $f = \frac{1}{2}\mathbb{1}_{x=0} + \mathbb{1}_{x>0}$. This is equal to its average on small balls, so it is certainly subharmonic on average, but it's far from being a subharmonic function. For example, $f(1) = 1$, but $A_f(1;2) = 3/4 < 1$.

What has gone wrong? This is not a very regular function: the Laplacian is $h \mapsto -h'(0)$, which is not even a signed measure. We will need a certain amount of regularity to get useful information.

This example also shows that we must be careful: convolutions of subharmonic on average functions with continuous functions are not necessarily subharmonic, even if the continuous function has a small support.

[1] Exercise: prove that we do really have $|\frac{1}{2s}\int_{-s}^{s} f(y)\,dy - f(0)| \leq Cs^2/6$ if $|f''| \leq C$.

5.8 If the Laplacian is a signed measure, and subharmonic on average on a measurable set, then the signed measure is positive on that set

If f is regular enough that $\nabla^2(f\,d\lambda)$ is a signed measure, then we can get a very precise result about subharmonicity on average.

Theorem 14. *Suppose $\nabla^2(f\,d\lambda) = d\nu$ where ν is a signed measure. If f is limit-subharmonic on a measurable set E, then E is a positive set for ν.*

Proof. Suppose E is not positive. Let E' be a subset of E with $\nu(E) < 0$. By Lemma 15 (which we postpone until later), $\exists x \in E'$ with

$$\limsup_{t\to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} = -c < 0.$$

Let $s > 0$ be small enough that $\nu(B_t(x))/\lambda(B_t(x)) < -c/2$ for $t < s$ and the subharmonic inequality holds for $B_s(x)$.

Lemma 11 and Corollary 12 tell us that, for each $0 < t < s$, there is a nonnegative radially symmetric continuous function $h_{s,t}$ with

$$\int h_{s,t}(y)\,\nu(dy) = A_f(x;s) - A_f(x;t).$$

Also, $h_{s,t}$ is supported on B_s, radially symmetric, and it's decreasing as y gets farther from x. So $\{y : h_{s,t}(y) > \alpha\}$ is a ball of radius $\le s$ around x, and

$$\begin{aligned}\int h_{s,t}\,d\nu &= \int_0^s \nu\{h_{s,t} > \alpha\}\,d\alpha \\ &\le -\frac{c}{2}\int_0^s \lambda\{h_{s,t} > \alpha\}\,d\alpha \\ &= -\frac{c}{2}\int h_{s,t}\,d\lambda = -\frac{c}{4(d+2)}(s^2 - t^2).\end{aligned}$$

In the last step we have used Lemma 12. Take $t \to 0$ and use the fact that $A_f(x;t) \to f(x)$ as $t \to 0$, because f is a limit of radial averages at $x \in E$.

$$-\frac{c}{4(d+2)}s^2 \ge \limsup_{t\to 0} \int h_{s,t}\,d\nu = A_f(x;s) - f(x) \ge 0$$

which is a contradiction. Therefore, E is positive. □

Here is the lemma we are owed:

Lemma 15. *If $E \subseteq \Omega$ is measurable and ν is a signed measure with $\nu(E) < 0$, then there is a point $x \in E$ with*

$$\limsup_{t\to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} < 0.$$

Proof. Let $\mu = |\nu| + \lambda$, and $f = d\nu/d\mu$. Then $\int f\, d\mu = \nu(E) < 0$, so $f < 0$ on a set of μ-positive measure.

By the Lebesgue-Besicovitch differentiation theorem,

$$\lim_{t \to 0} \frac{\nu(B_t(x))}{\mu(B_t(x))} = f(x)$$

except on a set N with $\mu(N) = 0$. And $\{f < 0\}$ has μ-positive measure, so there is some point x in $\{f < 0\} \cap N^c$.

For that point, we have $\nu(B_t(x))/\mu(B_t(x)) \to f(x) < 0$, so $\nu(B_t(x))$ is negative for small enough t, and $\lambda \le \mu$, so

$$\limsup_{t \to 0} \frac{\nu(B_t(x))}{\mu(B_t(x))} \le \limsup_{t \to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} < 0.$$

That proves the result. □

Note. The regularity is important for this lemma: *if* $\nabla^2(f\, d\lambda)$ is a signed measure, *then* it's positive on the set where it's subharmonic on average.

For the example before, if we set $f = \frac{1}{2}\mathbb{1}_{x=0} + \mathbb{1}_{x>0}$ again, the function is subharmonic on average everywhere, but $\nabla^2(f\, d\lambda) : h \mapsto -h'(0)$ isn't positive (and so it isn't a signed measure).

Exercise. What exactly breaks down when $\nabla^2(f\, d\lambda)$ isn't a signed measure? How much of the proof still works for $f = \frac{1}{2}\mathbb{1}_{x=0} + \mathbb{1}_{x>0}$?

5.9 On an open set, positivity implies subharmonicity

Now we will go in the other direction, from information about the distribution to subharmonicity.

Lemma 16. *Let f be locally integrable on an open set Ω. Suppose $\nabla^2(f\, d\lambda)$ is positive on Ω. Then there is a subharmonic $\bar{f}$ on Ω with $\bar{f} = f$ a.e. on Ω.*

Proof. Let $x \in \Omega$ and $t < s < d(x, \Omega^c)$. If $h_{s,t}$ is as in Theorem 14, then Theorem 11 tells us that $A_f(x; s) - A_f(x; t) = \nabla^2(f\, dx)(h) \ge 0$, so $A_f(x; t)$ decreases to a limit (possibly $-\infty$) as $t \to 0$. Let $\bar{f}$ be that limit:

$$\bar{f}(x) := \lim_{s \to 0} A_f(x; s) = \inf_{s > 0} A_f(x; s).$$

The Lebesgue differentiation theorem tells us that $A_f(x; s) \to f(x)$ for almost every x, so $f = \bar{f}$ almost everywhere. We claim $\bar{f}$ is subharmonic on Ω.

For $0 < s < d(x, \Omega^c)$,

$$\bar{f}(x) \le A_f(x; s) = A_{\bar{f}}(x; s),$$

so $\bar{f}$ satisfies (b) in the definition of subharmonicity.

We prove that it also satisfies (a), upper semicontinuity. Let $x_n \in \Omega$ let $r > 0$ with $B_{2r}(x) \subseteq \Omega$. Eventually $x_n \in B_r(x)$. Decompose

$$\bar{f}(x_n) = \Big[\bar{f}(x_n) - A_f(x_n; r)\Big] + \Big[A_f(x_n; r) - A_f(x; r)\Big] + A_f(x; r).$$

The first summand is nonpositive by definition. The second one converges to 0 as $n \to \infty$, because $A_f(\cdot; r)$ is continuous where it is defined.[2] Take the lim sup of both sides as $n \to \infty$ and then take $r \to 0$ to get $\limsup \bar{f}(x_n) \leq \bar{f}(x)$. So $\bar{f}$ is upper semicontinuous also and therefore subharmonic. □

6 Solving the quadrature set problem

From now on, suppose w is a bounded, properly supported weight function.[3]

We will prove the existence of a quadrature set for w. We start by posing a minimization problem, extract the quadrature set from the solution, and then prove that it is a quadrature set.

Definition. If ψ and ψ' are distributions, then we say that $\psi \leq \psi'$ if $\psi' - \psi$ is a positive distribution.

Problem. Within the class of functions that are limits of radial averages on all $\mathbb{R}^d$, find the smallest nonnegative f with $\nabla^2(f\, d\lambda) \leq (1 - w)\, d\lambda$.

We'll find a function that's less than or equal to than any other such function pointwise, so we don't need to be specific about "smallest."

Any function can be set to zero on a null set without changing the distributional Laplacian, so without at least a weak continuity condition, no function can be minimal except for zero, which is only a solution if $w \leq 1$ a.e.

6.1 Intuitive justification for this problem

Suppose we are in $d = 2$ for ease of imagination. Arrange clay on a metal plate so that the height of the clay at x is $w(x)$. Then crush the clay down to height 1 by applying pressure to it from above. The result is a new thin sheet of clay of roughly uniform height and a certain shape Q.

Let $f(x)$ be the 'total pressure' at x. Suppose the total net flow of clay through a point is (proportional to) the gradient of the pressure, $-\nabla f(x)$. By conservation of mass, $\nabla \cdot (-\nabla f(x)) = -\nabla^2 f(x)$ will be the total decrease in the level of clay at x. The final level of clay cannot be more than 1, so we have the inequality $w + \nabla^2 f \leq 1$ or $\nabla^2 f \leq 1 - w$.

When we push on the clay, it flows out radially, with no preferred direction. Let s be any integrable subharmonic function on the final shape Q. When the clay is pressed down at a point within Q, the clay moves out radially, and subharmonic functions are less than or equal to their averages on balls in Q, so the integral of s against the mass distribution does not decrease.

We ask for the smallest nonnegative f because the total pressure can't be negative, but we want to push the clay as little as possible in the hope that

[2]This follows from general properties of convolutions, or it can be proven directly by writing $A_f(r; x_n) - A_f(r; x) = \int (\mathbb{1}_{B_r(x_n)} - \mathbb{1}_{B_r(x)})\, f\, dy$ and observing that the integrand converges pointwise to zero almost everywhere and is dominated by $|f|$.

[3]Recall from Section 2.4 that a weight function is properly supported if it is at least 1 on an open set Ω and identically zero outside Ω.

the set where pressure is applied, $\{f > 0\}$, will be contained in the final shape of clay. If all pressure is applied within Q, then the integral of s against the mass distribution never decreases as we go from $w(x)$ to $\mathbb{1}_Q$, so we get the quadrature set property, $\int s(x)w(x)\,dx \leq \int s(x)\mathbb{1}_Q(x)\,dx$.

We will see that this nonsense really works, and $Q := \{f > 0\} \cup \{w \geq 1\}$, the area where we applied pressure plus the area where there was already clay on the table at the start, is indeed a quadrature set.

6.2 Newtonian potentials

If w is a bounded, compactly supported weight function, let the *Newtonian potential* of w be the convolution of w with the free-space Green's function:

$$Nw(y) := \int_{\mathbb{R}^n} G(x,y)w(x)\,dx.$$

This is really a convolution because $G(x,y)$ is a function of $x - y$ only.

The convolution of a bounded function with an integrable function is continuous, so $Nw \in C(\mathbb{R}^n)$, and the first derivative is continuous also:

$$\frac{\partial Nw}{\partial y_i}(y) = \frac{\partial}{\partial y_i}\int_{\mathbb{R}^n} G(x,y)w(x)\,dx = \int_{\mathbb{R}^n} \frac{\partial G}{\partial y_i}(x,y)w(x)\,dx.$$

Green's function is symmetric in the variables, so if $h \in C_c^\infty$, then

$$\begin{aligned}\int Nw(y)\nabla^2 h(y)\,dy &= \int w(y)N\nabla^2 h(y)\,dy \\ &= \int -h(x)w(x)\,dx\end{aligned}$$

by Lemma 1. Therefore $\nabla^2 Nw = -w$ for any such weight function w.

We can make this a bit more general. If w is bounded and *constant* outside a compact set, let c be the constant, and define $Nw := N(w-c) + \frac{c}{2d}|x|^2$. Again $\nabla^2(-Nw) = w$.

6.3 Minimization over superharmonic functions

We use Newtonian potentials to transfer the minimization problem to the theory of superharmonic functions.

For the sake of completeness:

Definition. If f is a function, it's *superharmonic* if $-f$ is subharmonic.

The theorems about subharmonic functions carry over to superharmonic functions with a minus sign. In particular, we have these three statements:

Theorem −5. *If f is a superharmonic function on an open set Ω, then $\nabla^2 f$ is a negative distribution.*

Lemma −13. *A superharmonic function is a limit of radial averages.*

Corollary −16. *If the Laplacian of f is a negative distribution on an open set, then f is equal almost everywhere to a superharmonic function.*

This is enough to turn the minimization problem into a question about superharmonic functions.

Theorem 17. *The minimization problem is equivalent to:*

> *Find the smallest nonnegative function f on $\mathbb{R}^d$ with the property that the sum $f + N(1-w)$ is superharmonic everywhere in $\mathbb{R}^d$.*

Proof. The minimization problem asks us to find the smallest function $f \geq 0$ which is a limit of radial averages everywhere with $\nabla^2 f \leq 1 - w$. We will show that the two classes of functions are the same: f is a limit of radial averages with $\nabla^2 f \leq 1 - w$ if and only if $f + N(1-w)$ is superharmonic.

Suppose f is a limit of radial averages and $\nabla^2 f \leq 1 - w$. Then

$$\nabla^2[f + N(1-w)] = \nabla^2 f - 1 + w \leq 0.$$

By Corollary −16, $f + N(1-w) = \bar{f}$ a.e. for some superharmonic $\bar{f}$.

Averaging is linear, so $f + N(1-w)$ is a limit of radial averages, and so is the superharmonic function $\bar{f}$ by Lemma −13. They are equal almost everywhere, so $A_{f+N(1-w)}(x;r) = A_{\bar{f}}(x;r)$ for every choice of $x \in \mathbb{R}^d$ and $r > 0$. Take $r \to 0$ to see that $f + N(1-w) = \bar{f}$ is indeed superharmonic.

Suppose $f + N(1-w)$ is superharmonic. By Theorem −5,

$$\nabla^2 f - 1 + w = \nabla^2\big[f + N(1-w)\big] \leq 0,$$

and $\nabla^2 N = -\mathrm{id}$, so $\nabla^2 f \leq 1 - w$. By Lemma −13, $f + N(1-w)$ is a limit of radial averages, and $N(1-w)$ is continuous, so by linearity f is a limit of radial averages with $\nabla^2 f \leq 1 - w$.

The two conditions are therefore equivalent. □

We can now use the fundamental convergence theorem for superharmonic functions to find a minimum.

Theorem 18 (Fundamental convergence theorem)**.**
Let Γ be a family of superharmonic functions defined on an open subset of $\mathbb{R}^d$ and locally uniformly bounded below. Let $u(x)$ be the pointwise infimum of all the functions in Γ. Let $u_+(x) = \min\{u(x), \liminf_{y\to x} u(y)\}$.
Then $u_+ = u$ almost everywhere, and u_+ is superharmonic.

Proof. See for example Section 1.III.3 of Doob [1].

Corollary 19. *Let π be a measurable, bounded function on $\mathbb{R}^d$ that is constant outside a compact set. Then there is a smallest $f \geq 0$ with the property that $f + N\pi$ is superharmonic.*

Proof. Let $\Gamma = \{u : u \text{ is superharmonic}, \gamma \geq N\pi\}$. The functions in this class are uniformly bounded below on any compact set K by $\min_K N\pi$.

Apply the fundamental convergence theorem to Γ to get a superharmonic function u_+ less than or equal to every function in Γ. Then $u_+ \in \Gamma$:

$$\begin{aligned} u_+(x) &= \min\{u(x), \liminf_{y \to x} u(y)\} \\ &\geq \min\{N\pi(x), \liminf_{y \to x} N\pi(y)\} \\ &\geq N\pi(x) \end{aligned}$$

by continuity of $N\pi(x)$, so it obeys the inequality, and it's superharmonic.

Set $f := u_+ - N\pi$. Then $f \geq 0$ and $f + N\pi$ is superharmonic, and if that is true for g, then $g + N\pi \in \Gamma$ and $f + N\pi = u_+ \leq g + N\pi$, so $f \leq g$. □

Corollary 20. *There is a smallest function $f \geq 0$ that is a limit of radial averages and satisfies $\nabla^2(f\,dx) \leq (1-w)\,d\lambda$.*

Proof. Combine Corollary 19 with Theorem 17. □

In the next section, we will characterize the Laplacian of the minimal function, and discover that there is a quadrature set hiding inside it.

6.4 Finding the Laplacian $\nabla^2(f\,dx)$

Suppose w is a properly supported weight function. Let f be the function promised by Corollary 20. Then $\nabla^2(f\,dx) - (1-w)\,d\lambda$ is a negative distribution, so by Corollary 6, it must be $-d\mu$ for some locally finite measure μ. Therefore $\nabla^2(f\,dx)$ is locally a signed measure ν with $d\nu = (1-w)\,d\lambda - d\mu$.

Let $A := \{f > 0\}$. Then $d\mu = (1-w)\mathbb{1}_{A^c}\,d\lambda$ and $d\nu = (1-w)\mathbb{1}_A\,d\lambda$, which we prove by dividing the space up into three pieces: the original set, the complement of its closure, and the topological boundary between them.

Lemma 21. *On the open set $A = \{f > 0\}$, $d\mu = 0$ and $d\nu = (1-w)\,d\lambda$.*

Proof. Let $\gamma = f + Nu$ as before, so $\nabla^2(\gamma\,d\lambda) = -d\mu$. Let $x \in A$, so $f(x) > 0$ and $\gamma(x) > Nu(x)$. In that inequality, the left-hand function is superharmonic and the right-hand one is continuous, so both are limits of radial averages. Choose a small radius $r > 0$ so that

$$\min_{y \in B_r(x)} \gamma(y) > \max_{y \in B_r(x)} Nu(y).$$

Let

$$\gamma' = \begin{cases} \gamma & \text{outside } B_r(x) \\ \text{Poisson integral of } \gamma|_{\partial B_r(x)} & \text{on } B_r(x). \end{cases}$$

It's well-known that this is still a superharmonic function, and it's less than or equal to γ. Also $\gamma' \geq Nu$, because, by the choice of r,

$$\gamma'(y) \geq \min_{y \in B_r(x)} \gamma(y) > Nu(y)$$

for y in the ball $B_r(x)$, and $\gamma'(y) = \gamma(y) \geq Nu(y)$ for all other y. But γ is the minimal superharmonic function with $\gamma \geq Nu$, so $\gamma = \gamma'$.

In particular γ is harmonic on the ball $B_r(x)$. Therefore $\nabla^2\gamma = 0$ on that ball, and $d\nu = \nabla^2 f = -\nabla^2 Nu = (1-w)\,d\lambda$. That is true on a ball on a neighbourhood of every point $x \in A$, so it's true on all of A. $\square$

The second piece is trivial:

Lemma 22. *On the open set $\overline{A}^c$, $d\nu = 0$.*

Proof. The function f is identically zero there, so $\nabla^2(f\,dx) = 0$ on $\overline{A}^c$. $\square$

The tricky part is the boundary ∂A. We can use the Lebesgue density theorem to get rough bounds on $\nu|_{\partial A}$:

Theorem 23. *If $E \subseteq \partial A$, then $0 \leq \nu(E) \leq \lambda(E)$.*

Proof. We have $\nabla^2 f \leq (1-w)\,d\lambda$, so $d\nu \leq d\lambda$. That's the upper bound.

By definition, $f + N(1-w)$ is superharmonic, and $N(1-w)$ is continuous, so f is a limit of radial averages. If x is any point in ∂A, then

$$f(x) = 0 \leq \frac{1}{\lambda(B_r)} \int_{B_r(x)} f(y)\,dy.$$

Therefore, f is subharmonic on average on ∂A. Use Corollary 14 on $-f$ to get $\nabla^2 f = d\nu \geq 0$ on ∂A. That's the lower bound. $\square$

We summarize the last three theorems:

Corollary 24. *The measure ν is absolutely continuous with respect to μ on $\mathbb{R}^d$, and $\nu = (1-w)\lambda$ on A, $\nu = 0$ on $\overline{A}^c$, and $0 \leq \nu \leq \lambda$ on ∂A.*

Let $\rho := d\nu/d\mu$ be some version of the Radon-Nikodym derivative with $\rho = 1 - w$ on A and $\rho = 0$ on $\overline{A}^c$. If ∂A is a set of measure zero, then we already know ρ almost everywhere, so we know $\nabla^2 f = \rho\,d\lambda = (1-w)\mathbb{1}_A\,d\lambda$. If not, we must do more work.

Let's start by proving a lemma that says that bounds on ρ give us strong bounds on f near its zeroes.

Lemma 25. *Suppose $\nabla^2 f = \rho\,d\lambda$ where $0 \leq \rho \leq 1$, and $f \geq 0$.*

If $f(x) = 0$, then $f(y) \leq \max|\rho|\,O(|y-x|^2)$ where the implicit constant in the O-notation depends only on the dimension d.

Let $A_f(y;r)$ denote the average of f on $B_r(y)$, as in Lemma 16. Choose $t < s$. Corollary 11 tells us that $A_f(y;s) - A_f(y;t) = \nabla^2(f\,dx)(h) = \int h\rho\,d\lambda$ for a certain $h \geq 0$, and by Corollary 12, $\int h\,d\lambda = (s^2 - t^2)/2(d+2)$. So,

$$|A_f(y;s) - A_f(y;t)| \leq \left| \int h\,d\nu \right| \leq \max|\rho| \int h\,d\lambda = \frac{\max|\rho|}{2(d+2)}(s^2 - t^2).$$

Take $t \to 0$ to get $|A_f(y;s) - f(y)| \leq \max|\rho|\,O(s^2)$.

We can use the estimate to compare $A_f(y;s)$ and $f(y)$, and we can also use it to estimate $A_f(x;2s) = |A_f(x;2s) - f(x)| = O(s^2)$. Then

$$A_f(y;s) = \frac{1}{\lambda(B_s)} \int_{B_s(z)} f(z)\,dz \le \frac{1}{\lambda(B_s)} \int_{B_{2s}(x)} f(z)\,dz = 2^d A_f(x;2s),$$

so $f(y) = A_f(y;s) + O(s^2) \le 2^d A_f(x;2s) + O(s^2) \le O(s^2)$. □

Theorem 26. *If $\nabla^2 f = d\nu = \rho\,d\lambda$ with $0 \le \rho \le 1$ and $f \ge 0$, and $f \equiv 0$ on the complement of an open set A, then $\rho = 0$ almost everywhere on ∂A.*

Proof. Suppose not: $\nu(\partial A) > 0$. Then the Lebesgue measure of the boundary ∂A must certainly be positive. By the Lebesgue density theorem,

$$\lim_{r\to 0} \frac{\lambda(B_r(x) \cap \partial A)}{\lambda(B_r(x))} = 1$$

for almost every point in ∂A. Let N be the null set of points for which that isn't true, and fix $x \in \partial A \setminus N$. Then $\lambda(B_r(x) \cap A) = o(1)\,\lambda(B_r)$ as $r \to 0$, although the constant in the o-notation depends on the point x.

By the last theorem, $f(y) = O(|y-x|^2)$, but $f = 0$ on A^c, so the only contributions to the average $A_f(x;s)$ are from the points in $B_r(x) \cap A$. Therefore

$$A_f(x;s) = \frac{1}{\lambda(B_s)} \int_{B_s(x)\cap A} f(y)\,dy = \frac{\lambda(B_s(x) \cap A)}{\lambda(B_s)} O(s^2) = o(s^2)$$

for $x \in \partial A \setminus N$. But $\int_{\partial A} \rho(y)\,dy = \nu(\partial A) > 0$, so by the ordinary Lebesgue differentiation theorem applied to ρ, there is a set of full measure in ∂A with

$$\liminf_{r\to 0} \frac{\nu(B_r(x))}{\lambda(B_r)} \ge \liminf_{r\to 0} \frac{1}{\lambda(B_r)} \int_{B_r(x)\cap\partial A} \rho(y)\,dy > 0.$$

Let c be that lim inf. For small r, we must have $\nu(B_r(x))/\lambda(B_r) > c/2$. Pick $t \in (0,s)$, construct the function χ in Lemma 11, and repeat the reasoning in the proof of Theorem 14 to get the inequality $\int \chi\,d\nu \ge (c/2)\int \chi\,d\lambda$.

Then the left-hand side of that inequality is $A_f(x;s) - A_f(x;t)$ by Lemma 11, and the right-hand side is $\frac{c}{4(d+2)C_d}(s^2 - t^2)$ by Lemma 12, so

$$A_f(x;s) - A_f(x;t) \ge \frac{c}{4(d+2)C_d}(s^2 - t^2).$$

Take $t \to 0$ to get the lower bound $A_f(x;s) \ge (\text{positive constant}) \times s^2$.

This contradicts the estimate $A_f(x;s) = o(s^2)$ for $x \in \partial A \setminus N$, so our assumption was wrong, and the ν-measure of the boundary must be zero.

Corollary 27. *The Radon-Nikodym derivative ρ is equal a.e. to $(1-w)\mathbb{1}_A$. The Laplacian $\nabla^2(f\,dx) = d\nu = \rho\,d\lambda$ is $(1-w)\mathbb{1}_A\,d\lambda$.*

Proof. The theorem tells us that $\rho = 0$ almost everywhere on ∂A, and we've already chosen a version with $\rho = (1-w)$ on A and $= 0$ on $\overline{A}^c$. □

6.5 A quadrature set for Green's functions

We are now able to exhibit the quadrature set, although we're only about three-quarters of the way to the proof that it really is one.

In the last section, we discovered that, if f is a solution for the minimization problem, then $\nabla^2(f\,d\lambda) = (1-w)\mathbb{1}_A\,d\lambda$ where $A = \{f > 0\}$.

Theorem 28. *If f is as above, then $f = N\big[(w-1)\mathbb{1}_A\big]$.*

Proof. Denote $(w-1)\mathbb{1}_A$ by φ. Then

$$\nabla^2((f - N\varphi)\,d\lambda) = \nabla^2(f\,d\lambda) + (w-1)\mathbb{1}_A = 0.$$

Lemma 16 tells us that $f - N\varphi$ is harmonic[4]. We must show that it is zero, and we do that with Liouville's theorem.

First, A is bounded, because if $r := \max\{|x| : w(x) > 0\}$, one candidate for the minimization problem is

$$f_b(x) := N\left[(\max w)\mathbb{1}_{B_r} - \mathbb{1}_{B_{(\max w)^{1/d}r}}\right].$$

This has a suitable Laplacian, is nonnegative[5], and is zero outside $B_{(\max w)^{1/d}r}$, so $f \le f_b$ must also be zero outside that ball and $A \subseteq \{|x| < (\max w)^{1/d}r\}$.

Let $h \in C_c^\infty(\mathbb{R}^d)$, $h \equiv 1$ on $\overline{A}$. (This is allowed because $\overline{A}$ is bounded, by the last paragraph.) Then $\nabla^2 h = 0$ in A, and $\varphi \equiv 0$ outside A, so we get $0 = \int f\,\nabla^2 h\,d\lambda = \nabla^2 f[h] = \int \varphi h\,d\lambda = \int \varphi\,d\lambda$, and the integral of φ is zero. As $x \to \infty$, $G(x,y) - G(x,0)$ converges to zero, so

$$N\varphi(x) = \int_{\mathbb{R}^d} G(x,y)\varphi(y)\,dy = \int_{\mathbb{R}^d}\Big[G(x,y) - G(x,0)\Big]\varphi(y)\,dy \to 0.$$

And f is zero outside A, so $f - N\varphi$ converges to zero as $x \to \infty$. It's harmonic, so it must be identically zero by Liouville's theorem. Therefore $f = N\varphi$. $\square$

Corollary 29. *Let $Q = A \cup \{w \ge 1\}$.*

Then $\varphi := (w-1)\mathbb{1}_A$ is equal to $w - \mathbb{1}_Q$ almost everywhere, and $N(\mathbb{1}_Q - w)$ is nonpositive everywhere and zero outside Q.

[4] It tells us that $f - N\varphi$ and $N\varphi - f$ are equal almost everywhere to subharmonic functions s and s', and all four functions are limits of radial averages, so both equalities are true everywhere: $f - N\varphi = s = -s'$, so $f - N\varphi$ is subharmonic and superharmonic.

[5] The average of Green's function $G(x;y)$ over a sphere $|y| = r$ is $G(x;0) = G(|x|\mathbf{e}_1;0)$ if $|x| > r$ and constant on the ball $|x| \le r$, because it's harmonic in both variables except at $x = y$. Green's function isn't singular enough for there to be a discontinuity when x is moved through the sphere, so the constant must be $G(r\mathbf{e}_1;0)$. Therefore

$$N[\mathbb{1}_{B_r(0)}](x) = \int_{B_r(0)} G((|x| \vee |y|)\mathbf{e}_1;0)\,dy.$$

It follows from this that $N[(\max w)\mathbb{1}_{B_r(0)}] = N[\mathbb{1}(B_{(\max w)^{1/d}r})] = \int_{B_r} G(x;0)\,dy$ outside the ball of radius $(\max w)^{1/d}r$, so $f_b = 0$ in that region. Green's function decreases with increasing radius, and some thought about how the integrals change shows that the first potential is strictly larger everywhere inside the ball. Therefore $f_b > 0$ there.

Proof. We know $1-w \geq 0$ in the sense of distributions on A^c, by Theorem so $w \leq 1$ almost everywhere. But w never takes values between 0 and 1, so $\{w = 0\}$ and $\{w = 1\}$ cover almost all of A^c, and

$$\begin{aligned} n &= \mathbb{1}_A - w + w\mathbb{1}_{A^c} \\ &= \mathbb{1}_A - w + \mathbb{1}_{\{w \geq 1\} \cap A^c} \text{ almost everywhere} \\ &= \mathbb{1}_Q - w. \end{aligned}$$

The two functions are equal a.e., so $N(w - \mathbb{1}_Q) = N\varphi = f$, and by definition f is nonnegative everywhere and is zero outside $A \subseteq Q$. □

This set Q is our quadrature set, and we'll prove that in the next section.

6.6 Extending to all integrable superharmonic functions

Here is the grand climax of this chapter, Sakai's Lemma 5.1 [6]. For completeness, we present the proof and various minor results that are used.

Theorem 30 (Sakai's Lemma 5.1)**.** *Let Q be an open bounded set. If there is a function $\varphi \in L^\infty(Q)$ with $N\varphi \geq 0$ on Q and $N\varphi = 0$ on Q^c, then*

$$\int s\varphi \, dy \leq 0$$

for every integrable subharmonic function s on Q.

Proof. The basic idea is approximation, but it's delicate and relies on a tight estimate of the modulus of continuity of $N\varphi$.

Let s be an integrable subharmonic function on Q. Let $s_n := s * \psi_n$ be as defined in Theorem 5. As before, these are smooth and subharmonic, and they are defined on the set $\{x : d(x, \Omega^c) > 1/n\}$. On any compact subset of Q, they converge to s from above and in L^1.

Let $h_j \in C_c^\infty(Q)$ be test functions $0 \leq h_j \leq 1$ converging pointwise to 1. We will choose them carefully later. Each sh_j is supported on some compact subset of Q, so $s_n h_j \to sh_j$ in L^1 on that subset. Therefore:

$$\int s\varphi \, dy = \lim_{j\to\infty} \int sh_j\varphi \, dy = \lim_{j\to\infty} \left[\lim_{n\to\infty} \int s_n h_j \varphi \, dy \right].$$

Using $\nabla^2 N = -\mathrm{id}$ and the fact that $s_n h_j$ is a test function,

$$\int s_n h_j \varphi \, dy = -\int s_n h_j \nabla^2 N\varphi \, dy = -\int \nabla^2(s_n h_j) N\varphi \, dy.$$

Our goal now is to show that this integral is $\geq -C_0/j$ if h_j and $C_0 > 0$ are chosen appropriately; then the double limit must be nonpositive.

A smooth subharmonic function has positive Laplacian, so $\nabla^2 s_n \geq 0$, and we are assuming that $N\varphi \geq 0$. So $(\nabla^2 s_n)h_j N\varphi \geq 0$, and we will have

$$\begin{aligned}\nabla^2(s_n h_j)N\varphi &= (\nabla^2 s_n)h_j N\varphi + 2(\nabla s_n \cdot \nabla h_j)N\varphi + s_n(\nabla^2 h_j)N\varphi \\ &\geq 2(\nabla s_n \cdot \nabla h_j)N\varphi + s_n \nabla^2 h_j N\varphi \\ &= 2\nabla \cdot (s_n \nabla h_j N\varphi) - 2s_n(\nabla h_j \cdot \nabla N\varphi) - s_n \nabla^2 h_j N\varphi.\end{aligned}$$

Integrate over $\mathbb{R}^d$. The leftmost term on the right-hand side is the divergence of a compactly supported differentiable function, so the integral is zero.

$$\int \nabla^2(s_n h_j)N\varphi\, dy \geq -\int 2s_n(\nabla h_j \cdot \nabla N\varphi)\, dy - \int s_n \nabla^2 h_j N\varphi\, dy. \tag{3}$$

We'll show that both those integrals are small enough that the bound holds.

A bound on $|\nabla N\varphi|$ *and* $N\varphi$. For $y \in \Omega$, let $\delta = d(y, \Omega^c)$, and pick a point $x \in \Omega^c$ with $d(y,x) = \delta$. By Lemma 34 below, there is $C > 0$ with

$$\big|\nabla N\varphi(y)\big| = \big|\nabla N\varphi(y) - \nabla N\varphi(x)\big| \leq C\delta \log 1/\delta$$

if $\delta = |y - x|$ is sufficiently small. That also gives us a bound on $N\varphi(y)$:

$$|N\varphi(y)| = |N\varphi(y) - N\varphi(x)| \leq \int_0^\delta Ct \log 1/t\, dt \leq \frac{C}{2}\delta^2 \log 1/\delta.$$

Choice of h_j. We now choose test functions so that $|\nabla h_j||\nabla N\varphi|$ will be small, using the following lemma. The key is that the reciprocal of the bound on $|\nabla N\varphi(y)|$, namely $1/(C\delta \log 1/\delta)$, has infinite integral near $\delta = 0$.

Lemma 31 (Hedberg, [3], Lemma 4)**.** *There is a sequence* h_j *of smooth test functions supported on* Ω *with* $0 \leq h_j \leq 1$ *everywhere,* $h_j \equiv 1$ *when* $d(x, \Omega^c) > 1/j$*, and if we denote* $\delta := d(y, \Omega^c)$*,*

$$\big|\nabla h_j(y)\big| \leq 1/(j\delta \log 1/\delta), \qquad \left|\frac{\partial^2}{\partial y_i \partial y_\ell} h_j(y)\right| \leq 3/(j\delta^2 \log 1/\delta).$$

Proof. Let $\psi(t) = 1/(t \log 1/t)$, so $|\psi'(t)| \leq 2/(t^2 \log 1/t)$ for $t < 1/e$.

Let $\eta_j \in C_c^\infty(0, 1/j)$ be chosen so that $0 \leq \eta_j(t) \leq \psi(t)/j$ and $|\eta_j'(t)| \leq |\psi'(t)|/j$, but so that $\int_0^{1/j} \eta_j(t)\, dt = 1$. This is possible because $\psi(t), \psi'(t)$ have infinite integral on $(0, 1/j)$. Let $\mathrm{H}_j(t) = \int_0^t \eta_j(\tau)\, d\tau$.

Let $\Delta_\Omega(y)$ be the smooth distance function from Stein §VI.2 Theorem 2 [7], quoted below as Lemma 35. It's comparable to the distance $\delta(y) = d(y, \Omega^c)$, in the sense that $c_1 d(y, \Omega^c) \leq \Delta_\Omega(y) \leq c_2 d(y, \Omega^c)$ for some constants c_1, c_2. As in the lemma, we may take $c_2 = 1$ and $B_\alpha = 1$ for $|\alpha| \leq 2$.

Set $h_j := \mathrm{H}_j(\Delta_\Omega(y))$. Then $|\nabla \Delta_\Omega| \leq 1$ and $|\partial^2 \Delta_\Omega / \partial x_i \partial x_l| \leq 1/\delta$, so (with some calculation) we see that the gradient of h_j is bounded in L^2-norm by $\psi(t)/j$ and each second derivative is bounded in absolute value by

$$\left|\eta_j(\Delta(y))\frac{\partial^2 \Delta_\Omega}{\partial x_i \partial x_l} + \eta_j'(\Delta(y))\frac{\partial \Delta_\Omega}{\partial x_i}\frac{\partial \Delta_\Omega}{\partial x_l}\right| \leq \frac{3}{j\delta^2 \log 1/\delta}.$$

□

Conclusion. We have $|\nabla h_j| \le 1/(j\delta \log 1/\delta)$ and $|\nabla^2 h_j| \le 3d/(j\delta^2 \log 1/\delta)$. And both functions are zero outside $\{x : d(x, \Omega^c) < 1/j\}$. Therefore,

$$|\nabla h_j||\nabla N\varphi| \le C/j \quad \text{and} \quad |\nabla^2 h_j||N\varphi| \le 3Cd/j$$

for large enough j, so both integrals in equation (3) are $O(1/j)$:

$$\begin{aligned}
\limsup_{n\to\infty} \left| \int 2s_n(\nabla h_j \cdot \nabla N\varphi)\, dy \right| &\le \limsup_{n\to\infty} \frac{2C}{j} \int_{h_j>0} |s_n|\, dy \\
&= \frac{2C}{j} \int_{h_j>0} |s|\, dy \\
\limsup_{n\to\infty} \left| \int s_n(\nabla^2 h_j N\varphi)\, dy \right| &\le \limsup_{n\to\infty} \frac{3Cd}{j} \int_{h_j>0} |s_n|\, dy \\
&= \frac{3Cd}{j} \int_{h_j>0} |s|\, dy.
\end{aligned}$$

Set $C_0 = (2+3d)C$. Then, as promised earlier, $\int s_n h_j \varphi\, dy \ge -C_0/j$.

Take $n \to \infty$ and then take $j \to \infty$ to see that

$$\int s\varphi\, dy = -\lim_{j\to\infty} \lim_{n\to\infty} \int \nabla^2(s_n h_j) N\varphi\, dy \le 0$$

by the earlier calculation. □

We therefore finally have the theorem:

Theorem 32. *$Q = \{f > 0\} \cup \{w \ge 1\}$ is a quadrature set for w.*

Proof. By Corollary 29, $N(w - \mathbb{1}_Q)$ is nonnegative and zero outside Q, so we can use Theorem 30 with $\varphi = w - \mathbb{1}_Q$. That tells us that, if s is any integrable subharmonic function on Q, then $\int s\varphi\, dy = \int s(w - \mathbb{1}_Q)\, dy \le 0$, which is exactly the quadrature set property. □

Corollary 33.
If w is bounded and properly supported, there is a quadrature set for it.

6.7 Denouement 1: log-Lipschitz continuity

We owe two lemmas that we must prove. First, a lemma about the modulus of continuity of the Newtonian potential, which we use to bound both the potential and its first derivative.

The lemma states, roughly, that $\nabla N\varphi$ is very close to being Lipschitz.

Lemma 34 (Günther, [2], §13)**.** *Suppose φ is bounded and measurable and zero outside a bounded set E. If $y, y' \in \mathbb{R}^d$ and $|y - y'| = \varepsilon$, then*

$$\left| \frac{\partial N\varphi}{\partial y_i}(y) - \frac{\partial N\varphi}{\partial y_i}(y') \right| = O\left(\varepsilon \log \frac{1}{\varepsilon}\right).$$

The constant in the O-notation depends only on $\max|\varphi|$ *and* $\operatorname{diam} E$*.*

Proof. Write both terms as derivatives of integrals, move them both under the same integral sign, and move the derivative inside the integral, to get

$$\left|\frac{\partial N\varphi}{\partial y_i}(y) - \frac{\partial N\varphi}{\partial y_i}(y')\right| = \left|\int_\Omega \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right]\varphi(x)\,dx\right|.$$

This is justified because $\partial G/\partial y_i$ is locally integrable and φ is bounded.

Let $A = \{x \in \Omega : |x-y| < 2\varepsilon\}$, and break the integral on the last line up into $\int_A$ and $\int_{E\setminus A}$. The first derivatives of $G(x,y)$ are $O(||x-y||^{1-d})$, so

$$\int_A \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right]\varphi(x)\,dx = O\left(\int_{B_{2\varepsilon}(y)} ||x-y||^{1-d}\,dx\right) = O(\varepsilon).$$

For the part outside of A, we estimate the integrand with derivatives. By the mean value theorem, there is a point y'' on the line segment between y and y' with $\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y') = (y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')$. The second derivatives of $G(x;y)$ are $O(||x-y||^{-d})$, so that dot product is at most

$$|y-y'| \times \left|\nabla\frac{\partial G}{\partial y_i}(x,y'')\right| = O(\varepsilon||x-y''||^{-d}).$$

When x is not in A, $||x-y|| \geq 2\varepsilon$, so

$$\begin{aligned} ||x-y''|| &\geq ||x-y|| - ||y-y''|| \\ &\geq ||x-y|| - ||y-y'|| \geq \tfrac{1}{2}||x-y||. \end{aligned}$$

Therefore $||x-y''||^{-d} \leq 2^d||x-y||$ and $O(||x-y''||^{-d}) = O(||x-y||^{-d})$, and we have the more concrete bound $|(y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')| = O(\varepsilon||x-y||^{-d})$, where the bound depends only on $\max|\varphi|$.

If $E\setminus A$ is empty, the integral over it is zero. Otherwise, let $\overline{B_{r_1}}\setminus B_{r_0}$ be the minimal closed annulus containing $E\setminus A$, i.e. $r_0 := \inf\{|x-y| \in E\setminus A\}$ and $r_1 := \sup\{|x-y| : x \in E\setminus A\}$, with $r_0 \geq 2\varepsilon$ and $r_1 \leq \operatorname{diam} E$.

We can estimate the integral over $E\setminus A$ by

$$\begin{aligned} \int_{E\setminus A} \left|(y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')\right|\varphi(x)\,dx &= \int_{E\setminus A} O(\varepsilon||x-y||^{-d})\,dx \\ &\leq O\left(\int_{r_0}^{r_1} \varepsilon r^{-d}\,C_d r^{d-1}\,dr\right) \\ &= O\left(\varepsilon\log\frac{r_1}{r_0}\right). \end{aligned}$$

The constants in the O-notation depend only on $\max|\varphi|$.

But we know that $r_1/r_0 \leq \operatorname{diam} E/2\varepsilon = O(1/\varepsilon)$, so $O(\varepsilon\log r_1/r_0) =$

$O(\log 1/\varepsilon)$. Combine the estimates for A and $E \setminus A$ to get the result:

$$\begin{aligned}\int_E \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right]\varphi(x)\,dx &= \int_A (\cdots)\varphi(x)\,dx + \int_{E\setminus A}(\cdots)\varphi(x)\,dx \\ &= O(\varepsilon) + O\left(\varepsilon \log\frac{1}{\varepsilon}\right) \\ &= O\left(\varepsilon\log\frac{1}{\varepsilon}\right).\end{aligned}$$

This is the result that we want. □

6.8 Denouement 2: a smooth approximation of the distance function

The second lemma that we owe is below. We used it in the proof of Lemma 31 to construct test functions with certain bounds on the first derivatives.

The lemma states that if $\Omega \subseteq \mathbb{R}^n$ is open, there is a smooth approximation to the distance function $y \mapsto d(y, \Omega^c)$ which has nicely controlled derivatives.

Lemma 35 (Stein §VI.2 Theorem 2 [7])**.**
Let Ω be an open set. There exists a function $\Delta_\Omega(x)$ on Ω such that

(a) $c_1 d(x,\Omega^c) \le \Delta(x) \le c_2 d(x,\Omega^c)$ *for* $x \in \Omega$

(b) $\Delta_\Omega(x)$ *is* C^∞ *in* Ω *and*

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\Delta_\Omega(x)\right| \le B_\alpha d(x,\Omega^c)^{1-|\alpha|}$$

where B_α, c_1, c_2 *are independent of* Ω.

Sketch of proof. Ω can be decomposed into disjoint cubes of side length 2^{-n} in such a way that the size of each cube is less than $\frac{1}{2}$ and more than $\frac{1}{8}$ of its distance from the boundary. If we scale up each cube slightly by some factor $1 < \beta < 2$, the result is a locally finite covering of Ω.

Pick $h \ge 0$ smooth so that h is 1 on the unit cube and 0 on the cube of size $\beta > 1$. Scale and translate this to get a function h_ω for each cube ω in the decomposition which is 1 on ω and 0 on a slightly larger cube around ω.

Let $\Delta_\Omega(x) = \sum_\omega \operatorname{diam}(\omega) h_\omega(x) / \sum_\omega h_\omega(x)$. This is locally finite and has the right properties.

Note. We can choose a function with $c_2 = 1$ and $B_\alpha = 1$ for $|\alpha| \le 2$ by applying the lemma and then scaling the resulting function down by $\max\{\max_{|\alpha|\le 2} B_\alpha, c_2\}$.

7 Definition of the sum

We have shown that if $w \geq 0$ is a bounded, properly supported weight function, then there is a quadrature set for w. That was Corollary 33.

We can use this fact to finally define the quadrature sum. If A and B are any two bounded open sets, then the weight function $\mathbb{1}_A + \mathbb{1}_B$ is bounded and properly supported, so it has a quadrature set.

Theorem 36. *Let A and B be bounded open sets. There is a unique bulky open set that is a quadrature set for $w = \mathbb{1}_A + \mathbb{1}_B$.*

Proof. Let $w = \mathbb{1}_A + \mathbb{1}_B$. This meets the criteria, because $A \cup B$ is bounded. Let C be a quadrature set for that weight; such a set exists and is essentially unique by Corollary 33 and Corollary 4. Let $C' \supseteq C$ be the unique bulky open set which is essentially equal to C. If s is integrable and subharmonic on C', then s must also be superharmonic on $C \subseteq C'$, so

$$\int_A s\,dx + \int_B s\,dx \leq \int_C s\,dx = \int_{C'} s\,dx.$$

Therefore C' is also a quadrature set for w.

No other bulky open set can be a quadrature set, because C' is the only one which is essentially equal to C. This proves the result. □

So the smash sum is well-defined.

7.1 The axioms

We will denote the Diaconis-Fulton smash sum by $A \oplus B$.

Theorem 37. *This sum satisfies all the axioms, and is always bulky.*

Proof. The bulkiness is obvious from the definition. We must check six axioms. Translation invariance, rotation invariance, and commutativity follow from the uniqueness of the bulky quadrature set and the invariance of integrals and superharmonicity and bulkiness under those operations.

For example, if $s \in S^+(C + x)$,

$$\int_{C+x} s\,d\lambda \leq \int_{A+x} s\,d\lambda + \int_{B+x} s\,d\lambda,$$

so $C + x$ is essentially equal to $(A + x) \oplus (B + x)$, and both sets are bulky, so they are really equal. Conservation of mass is easy: ± 1 is harmonic, so

$$\int_{A\oplus B} \pm 1\,d\lambda \leq \int_A \pm 1\,d\lambda + \int_B \pm 1\,d\lambda$$

and $\lambda(C) = \lambda(A) + \lambda(B)$.

Associativity. We show that the sum is associative, which for once is not trivial. Let A, B, C be bounded open sets. Then if $s \in S^+((A \oplus B) \oplus C)$,

$$\begin{aligned}\int_{(A\oplus B)\oplus C} s\,\lambda &\le \int_{A\oplus B} s\,d\lambda + \int_C s\,d\lambda \\ &\le \int_A s\,d\lambda + \int_B s\,d\lambda + \int_C s\,d\lambda.\end{aligned}$$

So $(A \oplus B) \oplus C$ is a quadrature set for $\mathbb{1}_A + \mathbb{1}_B + \mathbb{1}_C$. On the other hand, the set $A \oplus (B \oplus C)$ is a quadrature set for the same weight, so they're equal up to a set of measure zero by Corollary 4. They're bulky, therefore equal.

Monotonicity. Let A, B be bounded open sets. Then A and $A \oplus B$ are quadrature sets for $\mathbb{1}_A$ and $\mathbb{1}_A + \mathbb{1}_B$ respectively, and $\mathbb{1}_A \le \mathbb{1}_A + \mathbb{1}_B$, so Theorem 3 says A is essentially contained in $A \oplus B$. The latter set is bulky, so A is really contained in $A \oplus B$, and we have the first part of monotonicity.

Let A, B, C be bounded open sets with $A \subseteq C$. Then $A \oplus B$ and $C \oplus B$ are quadrature sets for $\mathbb{1}_A + \mathbb{1}_B \le \mathbb{1}_C + \mathbb{1}_B$, so again $A \oplus B \subseteq C \oplus B$. $\square$

That finishes the proof that the Diaconis-Fulton smash sum exists and satisfies all the axioms.

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