SUPPLEMENT: EXISTENCE AND UNIQUENESS OF QUADRATURE DOMAINS

CONTENTS

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1. Introduction

Our goal in this supplement is to prove the uniqueness and existence of quadrature domains for certain weight functions. The proofs follow Sakai [6].

Notation. Lebesgue measure is denoted by λ. If A and B are two sets, then $A \Delta B := (A \setminus B) \cup (B \setminus A)$. Two sets are *essentially equal* if $\lambda(A \Delta B) = 0$, and A is *essentially contained in* B if $\lambda(A \setminus B) = 0$.

If $A, B \subseteq \mathbb{R}^d$, then we set $d(A,B) = \inf\{|x-y| : x \in A, y \in B\}$.

1.1. **Subharmonic functions.**

Let $\Omega \subseteq \mathbb{R}^d$ be an open set.

Let $B_r(x)$ be the open ball $\{y \in \mathbb{R}^d : |y-x| < r\}$, and $B_r := B_r(0)$ be the open ball around zero. If $h : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is a locally integrable function, then let $A_h(x;r)$ be the average of the function on the ball $B_r(x)$:

$$A_h(x;r) = \frac{1}{\lambda(B_r)} \int_{B_r(x)} h(y)\, dy.$$

This integral is well-defined for sufficiently small r. We will say that a locally integrable function h is *subharmonic* at a point $x \in \Omega$ if:

(a) the function is upper semicontinuous at x, so $\limsup_{y \to x} h(y) \le h(x)$,
(b) and it is bounded above by its averages on small balls around x:

$$h(x) \le A_h(x;r) \qquad \text{for all sufficiently small } r.$$

If h is subharmonic at every $x \in E \subseteq \Omega$, then it is *subharmonic on E*. If $-h$ is subharmonic, then h is said to be *superharmonic*.

1.1.1. *Some properties of subharmonic functions.*

Let Ω be an open subset of $\mathbb{R}^d$. Then:

- If $h \in C^2(\Omega)$, then h is subharmonic on Ω if and only if $\nabla^2 h \ge 0$ on Ω.

If h is subharmonic on the whole set Ω, then:

- If $F \subset \Omega$ is compact, then $\max_{x \in F} h(x) < \infty$, and
$$\max_{x \in F} h(x) = \max_{x \in \partial F} h(x).$$
- If $r > 0$ is *any* radius such that $\overline{B_r(x)} \subset \Omega$, then $h(x) \le A_h(x;r)$.

If h is both subharmonic and superharmonic, then h is *harmonic*. Every harmonic function is not just continuous but smooth, and $\nabla^2 h \equiv 0$.

These statements still hold if we replace Ω by a smaller open set. We will later be able to say some things even for non-open sets.

1.2. **A diffusion relation between weight functions.** Say a *weight function* is a bounded, nonnegative measurable function on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let f and g be two weight functions. We will say that *f can diffuse to g through* Ω if $f \equiv g \equiv 0$ on Ω^c, and

$$\int hf\,dy \leq \int hg\,dy \tag{1}$$

for every integrable subharmonic function h on Ω. We write $f \ll_\Omega g$.

1.2.1. *What does it mean for one weight to diffuse into another?* Why do we refer to this relation as a "diffusion"? We will try to explain using ideas from probability theory. This section is not logically important to the rest of the paper, but it might provide some intuitive understanding.

We define a random walk. Choose a probability measure μ on Ω, and choose measurable $\rho_1, \rho_2, \ldots : \Omega \to [0,\infty)$ with $\rho_n(x) < d(x, \Omega^c)$ for $x \in \Omega$.

Let $X \sim \mu$, and let $Z_1, Z_2, \ldots$ be independent uniform points in the unit ball $\{x \in \mathbb{R}^d : |x| < 1\}$. We set $Y_0 := X$, and $Y_n := Y_{n-1} + \rho(Y_{n-1})Z_n$. The walk moves at different speeds depending on its location, but it's isotropic in the sense that the direction of each step is uniform, and it never leaves Ω. We will think of this as a discrete diffusion through Ω.

If h_+ is a nonnegative subharmonic function on Ω, then

$$\mathbb{E}[h_+(Y_n)] \leq \mathbb{E}[A_{h_+}(Y_n;\rho(Y_n))] = \mathbb{E}[h_+(Y_n + \rho(Y_n)Z_{n+1})] = \mathbb{E}[h_+(Y_{n+1})]$$

by Fubini-Tonelli, although some of the integrals may be $+\infty$. Chaining these inequalities together, we get $\mathbb{E}[h_+(Y_m)] \leq \mathbb{E}[h_+(Y_n)]$ for $m \leq n$ and any nonnegative subharmonic function h_+.

Suppose that the random variables Y_m, Y_n have bounded densities f_m, f_n. Let h be any integrable subharmonic function. If $h_N := h \wedge (-N)$, then $h_N + N$ is nonnegative and superharmonic, so we can use the above inequality to get

$$\begin{aligned}\int_\Omega h_N f_m\,dx &= \mathbb{E}[h_N(Y_m)] = \mathbb{E}[h_N(Y_m) + N] - N \\ &\leq \mathbb{E}[h_N(Y_n) + N] - N = \mathbb{E}[h_N(Y_n)] = \int_\Omega h_N f_n\,dx.\end{aligned}$$

Both functions are dominated by a constant times h, so we are justified in taking $N \to \infty$ and getting the inequality $\int_\Omega h f_m\,dx \leq \int_\Omega h f_n\,dx$ for every integrable subharmonic function h.

So far so good. If we have an isotropic walk as above that stays inside Ω, and two of the steps X_m and X_n have bounded densities f_m and f_n, then those densities are related by $f_m \ll_\Omega f_n$.

Now we make a conceptual leap into the darkness, and claim that this relation $\ll$ should be sufficient evidence of isotropic diffusion through Ω. Intuitively, if f and g are weight functions with $f \ll_\Omega g$, then there should be some sort of isotropic diffusion-like process X_t with $X_0 \sim f$ and $X_T \sim g$. This is the reason for the name.

1.3. **Quadrature domains.** Say that a function w is a *weight function* if it is bounded, nonnegative, and measurable. A *quadrature domain* for a weight function w is a bounded open set Ω so that we have $w \equiv 0$ outside Ω, and

$$\int hw\,dx \leq \int_\Omega h\,dx$$

for every integrable subharmonic function h on Ω.

That is, the measure $\mathbb{1}_\Omega\,d\lambda$ is a diffusion of $w\,d\lambda$ through Ω.

We can immediately observe that $\int hw\,dx = \int_\Omega h\,dx$ for any bounded harmonic function h on Ω. So the quadrature domain has measure exactly $\int w\,dx$,

$$\lambda(\Omega) = \int_\Omega 1\,d\lambda = \int w\,d\lambda,$$

and its centre of mass is the same too: $\int_\Omega x_i\,d\lambda = \int x_i w\,d\lambda$.

The moment of inertia of Ω is at least as large as the moment of w:

$$\int x^2 w\,d\lambda \leq \int_\Omega x^2\,d\lambda.$$

This agrees with the intuition that Ω is a more spread-out version of w.

Quadrature domains are not entirely unique. If Ω is a quadrature domain for a weight w, and $\Omega' \supsetneq \Omega$ is a larger open set with $\lambda(\Omega' \setminus \Omega) = 0$, then Ω' is also a quadrature domain for w. But we will see they are essentially unique.

2. Uniqueness of quadrature domains

We will prove in this section that if a weight function w has two quadrature domains Ω and Ω', then Ω is essentially equal to Ω'. First, we need a large supply of subharmonic functions.

2.1. Green's function.

2.1.1. *Motivation: What is Green's function?* Loosely speaking, if we have a differential operator $A = \sum_\alpha a_\alpha \partial^\alpha$, we say that a kernel function $g(x,y)$ is "Green's function" if the kernel map

$$K_g h(x) = \int h(y) g(x,y)\,dy$$

is a right inverse for A. In other words, we have $AK_g h = h$ for any nice h.

The function spaces are left deliberately vague. This is not a precise definition; it is a broad term for a class of similar objects.

We find Green's function for the operator $-\nabla^2$, where

$$\nabla^2 = \sum_i \partial_i \partial_i = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \right).$$

We derive Green's function from the properties of Brownian motion as on page 80, section 3.3 of Mörters and Peres [4]. We will produce two flavours: an "unrestricted Green's function," and a "restricted Green's function" on any bounded open set. Both will be subharmonic.

2.1.2. *The unrestricted Green's function for $d \geq 3$.* Let $B_x(t)$ be Brownian motion started at $x \in \mathbb{R}^d$. It is a random continuous curve in $\mathbb{R}^d$, and the distribution of $B_x(t)$ at a fixed time t is

$$\mathbb{P}[B_x(t) \in A] = \int_A \mathfrak{p}(t;x,y)\,dy,$$

where $\mathfrak{p}(t;x,y) := (2\pi t)^{-d/2} \exp(-|x-y|^2/2t)$ is the diffusion kernel.

This function $\mathfrak{p}$ has two important properties. First, it is an approximation to the identity as $t \to 0$, in the sense that if

$$\int_{\mathbb{R}^d} \mathfrak{p}(t;x,y) f(y)\,dy \to f(x) \qquad \text{as } t \to 0.$$

Second, it solves the heat equation

$$\frac{\partial \mathfrak{p}}{\partial t} = \frac{1}{2} \nabla^2 \mathfrak{p}.$$

Note the factor of $1/2$, which we must carry into the formulas below.[1]

Let the *unrestricted Green's function* for $d \geq 3$ be *half* the integral of the unrestricted kernel from $t = 0$ to $t = \infty$:

$$G(x,y) := \frac{1}{2} \int_0^\infty \mathfrak{p}(t;x,y)\,dt = \frac{1}{(d-2)C_d} |x-y|^{2-d}.$$

The interested reader is encouraged to do this integral and check that this is true. Here $C_d := 2\pi^{d/2}/\Gamma(d/2)$, which is the area of the unit sphere in $\mathbb{R}^d$.

Green's function is the density of the expected time that the Brownian motion $B_x(t)$ spends in a set, because

$$\int_A G^{(d)}(x,y)\,dy = \int_A \int_0^\infty \mathfrak{p}(t;x,y)\,dt\,dy = \int_0^\infty \mathbb{P}[B_x(t) \in A]\,dt.$$

and $\int_0^\infty \mathbb{P}[B_x(t) \in A] = \int_0^\infty \mathbb{E}[\mathbb{1}_{B_x(t) \in A}] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{B_x(t) \in A}\,dt\right]$.

It is clear from the explicit form above that $G(x,y)$ is smooth on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ and symmetric in its arguments, $G(x,y) = G(y,x)$.

[1] It is arguable that $-\frac{1}{2}\nabla^2$ would be a more natural operator than ∇^2, not just because it appears in this equation but also because of its role as kinetic energy in quantum mechanics with $\bar{h} = 1$. But everyone's used to the Laplacian already.

Let $G_x(y) := G(x,y)$. One can check that G_x is locally integrable, harmonic on the set $\mathbb{R}^d \setminus \{x\}$, and superharmonic everywhere.

This gives us a large selection of integrable subharmonic functions to use in inequality (1). If $x \in \Omega$, then we can use $-G_x$. If $x \notin \Omega$, then we can use both G_x and $-G_x$. This turns out to be all we need, by Lemma 4.15.

2.1.3. *The unrestricted Green's function for $d = 1,2$.* The integral above is infinite for $d = 2$, essentially because Brownian motion is recurrent there. We can still guess Green's function for two dimensions with the following informal argument. If $-\nabla^2$ is the inverse of the kernel operator, then $-\nabla^2 \int G(x,y) \mathbb{1}_{B_r(x)}\, dy$ should be $\mathbb{1}_{B_r(x)}$.

So we can make the following invalid computation:

$$\begin{aligned} 1 &= -\nabla^2 \int G(x,y) \mathbb{1}_{B_r}\, dy \\ &\overset{?}{=} -\int \nabla^2 G(x,y) \mathbb{1}_{B_r}\, dy \\ &\overset{?}{=} -\int_{|z|=r} \mathbf{n} \cdot \nabla G\, ds, \end{aligned}$$

where in the third line we use Stokes's theorem (although it is invalid to do that because ∇G is discontinuous at x). Here ds is $(d-1)$-dimensional surface area, and $\mathbf{n}$ is the outer normal.

The integral is over the sphere $|z| = r$, which has surface area $C_d r^{d-1}$. Brownian motion has radial symmetry, so we might as well assume that Green's function does, too. Then ∇G points radially, and its magnitude is constant; therefore, $1 = -C_d r^{d-1} \mathbf{n} \cdot \nabla G$.

This gives us a value for $\partial G / \partial r = -1/C_d r^{d-1}$. Now, we assume that this is true and integrate it to guess a formula for G. We get

$$\begin{aligned} G(x,y) &:= -\frac{1}{\pi} \log |x-y| && \text{in } d = 2 \\ G(x,y) &:= -|x-y| && \text{in } d = 1. \end{aligned}$$

In both cases we have chosen constants to make the formulas nice.

These functions also have the properties that we listed above: they are symmetric, smooth on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$, and G_x is locally integrable, superharmonic everywhere, and harmonic on $\mathbb{R}^d \setminus \{x\}$.

Here is another way of reaching this which does not require us to be so clever about integrals, called "differentiating on the parameter." We pretend that the dimension is real and write down the asymptotics of G as $d \to 2$.

$$G(x,y) = \frac{1}{(d-2)C_d} |x-y|^{2-d} \approx \frac{1}{(d-2)C_d} (1 + (2-d) \log |x-y| + \cdots)$$

The first term of this expansion becomes infinite as $d \to 2+$, but it is a constant. We subtract the constants out and define

$$G(x,y) := -\frac{1}{\pi}\log|x-y|.$$

The formula gives us the right answer for $d = 1$ as well, even though the derivation makes no sense in that dimension. In any case, we have some sort of formula, which will turn out to be Green's function in two dimensions.

2.1.4. *G is Green's function for $-\nabla^2$.* Let Ω be an open subset of $\mathbb{R}^d$. If $h : \Omega \to \mathbb{R}$ is smooth and supported on a compact subset of Ω, we say it is a *test function*.

Lemma 2.1. *If h is a test function, then $\int G(x,y)(-\nabla^2 h(y))\,dy = h(x)$.*

Sketch of proof. Let $G_x(y) := G(x,y)$. Let $\varepsilon > 0$. If $y \neq x$,

$$-G_x\nabla^2 h = -\nabla\cdot(G_x\nabla h - h\nabla G_x),$$

because G is twice differentiable on that set and $\nabla^2 G = 0$ there.

If we integrate on the whole space minus a small ball around x,

$$-\int_{\mathbb{R}^d\setminus B_\varepsilon(x)} G(x,y)\nabla^2 h(y)\,dy = \int_{|y-x|=r}\left(G_x\frac{\partial h}{\partial r} - h\frac{\partial G_x}{\partial r}\right)ds.$$

There are no other boundary terms because h is compactly supported.

The surface area of the sphere $|y-x| = \varepsilon$ is $C_d\varepsilon^{d-1}$. Green's function is small compared to this, $G(x,y) = o(1/|x-y|^{d-1})$, so the integral of $G_x\partial h/\partial r$ is $o(1)$. On the other hand, the integral of $-h\partial G_x/\partial r = h(y)/C_d\varepsilon^{d-1}$ will be the average of h on the small sphere. As $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B_\varepsilon(x)} G(x,y)(-\nabla^2 h(y))\,dy = h(x).$$

Green's function is locally integrable and h is compactly supported, so by dominated convergence, we get $\int G(x,y)(-\nabla^2 h(y))\,dy = h(x)$, which is what we were trying to prove. □

So for sufficiently smooth functions, convolution by Green's function is the inverse of $-\nabla^2$.

2.1.5. *Green's function of a bounded open set.* Let C be a bounded open subset of $\mathbb{R}^d$. Let $T_x := \min\{t \geq 0 : B_x(t) \notin C\}$ be the first time that $B_x(t)$ leaves C.

We will say that the *diffusion kernel restricted to C* is the continuous function $\mathfrak{p}_C : (0,\infty)\times C\times C \to [0,\infty)$ satisfying

$$\mathbb{P}[B_y(t) \in A \text{ and } t < T_x] = \int_A \mathfrak{p}_C(t;x,y)\,dy.$$

It is like the unrestricted diffusion kernel, except that the Brownian motion is killed when it leaves C. The fact that there is such a function is proven in the same section 3.3 of Mörters and Peres [4].

Set $G_C(x,y) = \int_0^\infty \mathfrak{p}_C(t;x,y)\,dt$, so that we have

$$\int_A G_C(x,y)\,dy = \int_0^\infty \mathbb{P}[B_y(t) \in A \text{ and } t < T_x]\,dt.$$

This is our function, the *Green's function of C*.

From the properties of $\mathfrak{p}_C$ and Brownian motion, we know that G_C is nonnegative and symmetric. Here are some other well-known properties:

- $y \mapsto G_C(x,y)$ is superharmonic on C and harmonic on $C \setminus \{x\}$.
- $G_C(x,y) > 0$ when x and y are in the same connected component of C.
- The function $y \mapsto G(x,y) - G_C(x,y)$ is harmonic on C.

If $d \geq 3$, then we have $\mathfrak{p}_C \leq \mathfrak{p}$, so $0 \leq G_C(x,y) \leq G(x,y)$. If $d = 2$, then Mörters and Peres, Lemma 3.37, tells us that $|G(x,y) - G_C(x,y)|$ is bounded by $(1/\pi)\log R/r$, where $r := d(x,\Omega^c)$ and $R := \inf\{r : B_r(x) \supseteq C\}$. In both cases, the function $y \mapsto G_C(x,y)$ is bounded above by a function that's integrable on C, so it is also integrable on C.

2.2. **The extended Green's function.** Writers who describe G_D as "the Green's function" should be condemned to differentiate the Lebesgue's measure using the Radon-Nikodym's theorem.

— Joseph Doob

Even so, there is a subharmonic extension of Green's function to $\mathbb{R}^d \setminus \{y\}$, which we refer to as *the extended Green's function*.

Theorem 2.2. *If C is a bounded open set with Green's function G_C, there is a nonnegative extension $G_C^e : C \times \mathbb{R}^d \to [0,\infty]$ so that:*

- *$G_C^e(x,y) = G_C(x,y)$ when $y \in C$.*
- *$G_C^e(x,\cdot)$ is subharmonic on $\mathbb{R}^d \setminus \{x\}$ and superharmonic on C.*
- *$G_C^e(x,\cdot)$ is zero at almost every point in $\mathbb{R}^d \setminus C$.*
- *$G_C^e(x,\cdot)$ is integrable.*

Reference. This is (c) of Doob's Theorem 1.VII.4 [1]. Our set C is bounded and $d \geq 2$, so it is Greenian and the theorem applies.

Note that polar sets have Lebesgue measure zero, so Doob's conclusion that the function is zero at "quasi every finite point," or in other words on the complement of a polar set, implies that it holds on a set of full measure.

The extension is integrable because it's equal to G_C on C and zero almost everywhere outside of it. □

The above theorem is true for *every* bounded open set, even if it has sharp cusps, an infinite number of small holes, or a boundary that has positive Lebesgue measure. This generality is very important in our setting.

2.3. **Monotonicity of quadrature domains.** We can immediately conclude that quadrature domains are essentially monotone.

Theorem 2.3. *If $w \le w'$ are two nonnegative measurable functions and C, D are quadrature domains for w and w', then C is essentially contained in D.*

Proof. Let E be a connected component of C. We will prove that $E \setminus D$ has zero measure. If it's empty, we are done. Otherwise, choose $x \in E \setminus D$. By the definition of a quadrature domain and the above properties of G^e_C, we have

$$\int_C G^e_C(x,y)\,dx \le \int G^e_C(x,y)\,w\,dx \le \int G^e_C(x,y)\,w'\,dx \le \int_D G^e_C(x,y)\,dx.$$

Therefore, $\int_{C\setminus D} G^e_C(x,y)\,dx \le \int_{D\setminus C} G^e_C(x,y)\,dx$. But the second integral is zero, because $G^e_C(x,\cdot)$ is zero almost everywhere on C^c. Therefore,

$$\int_{C\setminus D} G^e_C(x,y)\,dx = 0.$$

Green's function is strictly positive on E, so $E \setminus D$ must have zero measure. An open set in Euclidean space has only countably many components, so

$$C \setminus D = \bigcup_{E \text{ component of } C} E \setminus D$$

also has zero measure, and C is essentially contained in D. □

Corollary 2.4. *Quadrature domains are essentially unique.*

Proof. By the lemma, two quadrature domains for the same weight are essentially contained in each other, so their set difference has measure zero.

3. Positivity of the Laplacian

3.1. **Positive distributions.** Let Ω be an open subset of $\mathbb{R}^d$, and recall that a *distribution* on Ω is a continuous linear functional on the space of test functions on Ω.

(We recall from distribution theory that a linear functional ψ on the space of test functions is continuous if and only if, for every compact $F \subset \Omega$, the restricted map $\psi|_{C^\infty_c(F)} : C^\infty_c(F) \to \mathbb{R}$ is continuous in some $\|\ \|_{C^n(F)}$ norm. In particular, measures and locally integrable functions are distributions, and the space is closed under differentiation.)

Let $D'(\Omega)$ be the vector space of distributions. Let $\psi \in D'(\Omega)$. Then $\partial_i \psi$ is the distribution $h \mapsto -\psi[\partial_i h]$, and $\nabla^2 \psi$ is the distribution $h \mapsto \psi[\nabla^2 h]$.

If μ is a locally finite measure or a signed measure[2] we write the corresponding distribution $h \mapsto \int h\, d\mu$ as $d\mu$. In the same way, if f is locally integrable, we write the distribution $h \mapsto \int hf\, d\lambda$ as $f\, d\lambda$.

(This is a little different from the usual notation, where distributions from measures are written as μ and distributions from functions are written as f.)

A distribution ψ is *positive* if $\psi[h] \geq 0$ whenever $h \geq 0$, and it is *negative* if $\psi[h] \leq 0$ whenever $h \geq 0$. We write $\psi \leq \psi'$ if $\psi' - \psi$ is positive.

3.1.1. *The Laplacian of a subharmonic function.* We will now prove that the distributional Laplacian of a subharmonic function on Ω is a locally finite measure.

Theorem 3.1. *If f is subharmonic on Ω, then $\nabla^2(f\, d\lambda)$ is positive on Ω.*

Proof. Fix $h \in C_c^\infty(\Omega)$ with $h \geq 0$. Let h be supported on compact $K \subset \Omega$. Let φ be a positive mollifier, that is, an infinitely differentiable nonnegative function on $\mathbb{R}^d$ with $\varphi(x) \equiv 0$ for $|x| \geq 1$ and $\int \varphi\, dx = 1$.

Let $\varphi_n(x) := n^d \varphi(nx)$. Let $n > d(F, \Omega^c)$. Then $f_n := f * \varphi_n$ is defined and infinitely differentiable on a neighbourhood of K. It is also subharmonic:

$$\begin{aligned} f * \varphi_n(x) &= \int_{B_{1/n}} f(x-y)\varphi_n(y)\, dy \\ &\leq \frac{1}{\lambda(B_r)} \int_{B_{1/n}} \int_{B_r} f(x-y+z)\varphi_n(y)\, dz\, dy \\ &= \frac{1}{\lambda(B_r)} \int_{B_r} f * \varphi_n(x+z)\, dz. \end{aligned}$$

Therefore, $\nabla^2(f * \varphi_n)$ exists and is nonnegative.

It is well-known that $f * \varphi_n \to f$ in $L^1(F)$. So,

$$\begin{aligned} \nabla^2(f\, d\lambda)(h) = \int f \nabla^2 h\, dx &= \lim_{n\to\infty} \int (f * \varphi_n) \nabla^2 h\, dx \\ &= \lim_{n\to\infty} \int \nabla^2(f * \varphi_n)\, h\, dx \\ &\geq 0. \end{aligned}$$

That is true for every nonnegative $h \in C_c^\infty(\Omega)$, so $\nabla^2 f$ is positive on Ω. □

Lemma 3.2. *If ψ is a positive distribution on Ω, then there is a locally finite measure μ on Ω with $\psi(h) = \int h\, d\mu$ for every test function $h \in C_c^\infty(\Omega)$.*

Proof. See Rudin [5], chapter 6 exercise 4. We sketch the proof.

Let ψ be a positive distribution. If $K \subset\subset \Omega$ is a compact subset, then there is a nonnegative $h_1 \in C_c^\infty(\Omega)$ that is identically 1 on K. Positivity says

[2]Our signed measures are always bounded, $|\nu|(\Omega) < \infty$.

$0 \le \psi(h) \le \psi(h_1)$ if $h \in C_c^\infty(K)$ and $0 \le h \le 1$. Therefore, the restricted distribution $\psi|_K : C_c^\infty(K) \to \mathbb{R}$ is continuous with respect to $\|h\|_\infty$.

By the Riesz representation theorem and the positivity, there is a finite measure μ_K with $\psi(h) = \int h\, d\mu_K$ for every test function h supported on the compact set. These restricted measures are compatible, and we can combine them to get a locally finite measure μ on Ω with $\psi(h) = \int h\, d\mu$ for $h \in C_c^\infty(\Omega)$.

Corollary 3.3. *If f is subharmonic on Ω, then there exists a locally finite measure μ on Ω with $\nabla^2(f\, d\lambda) = d\mu$.*

Proof. The distributional Laplacian $\nabla^2(f\, d\lambda)$ is positive by Theorem 3.1, so there is a locally finite measure μ with $\nabla^2(f\, d\lambda) = d\mu$ by Lemma 3.2. □

3.1.2. *What is coming next.* Corollary 3.3 tells us a lot about functions which are subharmonic on open sets. What if a function f is subharmonic on a general measurable set?

It turns out that if we already know that the distributional Laplacian $\nabla^2(f\, d\lambda)$ is a signed measure $d\nu$, then we can get very precise information: if f is subharmonic on a measurable set E, then ν is positive on E.

That is a special case of Theorem 3.13. In the next few sections we will study the relationship between the spherical average function and the distributional Laplacian, and then finally prove that theorem.

3.2. **The existence of the spherical average function.** Suppose f is locally integrable, x is a point in Ω, and $0 < r < d(x, \Omega^c)$. Let the average on the sphere of radius r around x be

$$L_f(x; r) := \frac{1}{C_d} \int_{|z|=1} f(x + rz)\, dz.$$

Again, C_d is the total surface area of the unit sphere.

If the reader is doubtful about the notation $\int_{|z|=1} dz$, we offer a very concrete interpretation in the next section, Section 3.3.

Lemma 3.4. *Let Ω be an open set, and fix a point $x \in \Omega$.*

If f is locally integrable on Ω, then $L_f(x; r)$ is defined for almost every $r < d(x, \Omega^c)$, and $\int_0^s r^{d-1} |L_f(x; r)|\, dr < \infty$ for $s < d(x, \Omega^c)$.

Proof. Let $0 < s < R$. $\overline{B_s(x)}$ is compact, so $\int_{B_s(x)} |f|\, dx < \infty$.

Write this as a double integral in polar coordinates $y = x + rz$, where $r > 0$ and z is a point on the unit sphere:

$$\int_{B_s(x)} |f(y)|\, dy = \int_0^s \left[\int_{|z|=1} |f(x + rz)|\, r^{d-1}\, dz \right] dr.$$

The left-hand integral is finite, so Tonelli's theorem tells us that the integral in brackets is finite for a.e. $r \in (0,s)$, and therefore a.e. $r \in (0,R)$.

Therefore, $L_f(x;r) := \frac{1}{C_d}\int_{|z|=1} f(x+rz)\,dz$ is well-defined a.e., and

$$\begin{aligned}\int_0^s |L_f(x;r)|\, r^{d-1}\, dr &= \frac{1}{C_d}\int_0^s \left|\int_{|z|=1} f(x+rz)\right| r^{d-1}\, dz\, dr \\ &\le \frac{1}{C_d}\int_0^s \int_{|z|=1} |f(x+rz)|\, r^{d-1}\, dz\, dr \\ &< \infty.\end{aligned}$$

Corollary 3.5. *The function $r \mapsto L_f(x;r)$ is locally integrable on $(0, d(x,\Omega^c))$.*

Proof. If $0 < t < s < d(x,\Omega^c)$, then $(r/t)^{d-1} \ge 1$ when $r \in [t,s]$, so

$$\int_t^s |L_f(x;r)|\, dr \le \frac{1}{t^{d-1}}\int_t^s |L_f(x;r)|\, r^{d-1}\, dr < \infty.$$

Therefore, $r \mapsto L_f(x;r)$ is integrable on compact subsets of $(0, d(x,\Omega^c))$. □

Corollary 3.6. *The average of a locally integrable function f on $B_s(x)$ is*

$$A_f(x;s) = \int_0^s \frac{dr^{d-1}}{s^d} L_f(x;r)\, dr.$$

Proof. We use the change of variables $y = x + rz$ again to write

$$\begin{aligned}\int_{B_s(x)} f(y)\, dy &= \int_0^s \int_{|z|=1} f(x+rz)\, r^{d-1}\, dr\, dz \\ &= \int_0^s C_d L_f(x;r)\, r^{d-1}\, dr.\end{aligned}$$

Dividing this by the integral $\int_{B_s(x)} 1\, dy = C_d s^d/d$ gives the result. □

At this point we know that $L_f(x;r)$ makes sense and is integrable. The next step is to show that spherical averages are related to the distributional Laplacian by an integral equality. We do that in the next section.

3.3. **A digression: multidimensional polar coordinates.** In case one finds the "polar coordinates" $y = x + rz$ above to be a little suspicious, we will provide a concrete interpretation.

We define multidimensional polar coordinates $r, \varphi_1, \dots, \varphi_{d-2}, \theta$, where

$$\begin{aligned}
y_1 &= x_1 + r\sin\varphi_1,\\
y_2 &= x_2 + r\cos\varphi_1 \sin\varphi_2,\\
y_3 &= x_3 + r\cos\varphi_1 \cos\varphi_2 \sin\varphi_3,\\
&\vdots\\
y_{d-2} &= x_{d-2} + r\cos\varphi_1 \cdots \cos\varphi_{d-3} \sin\varphi_{d-2},\\
y_{d-1} &= x_{d-1} + r\cos\varphi_1 \cdots \cos\varphi_{d-3} \cos\varphi_{d-2} \sin\theta,\\
y_d &= x_d + r\cos\varphi_1 \cdots \cos\varphi_{d-3} \cos\varphi_{d-2} \cos\theta
\end{aligned}$$

The bounds are $r > 0$, $-\frac{\pi}{2} \le \varphi_1, \dots, \varphi_{d-2} \le \frac{\pi}{2}$, $0 \le \theta < 2\pi$. They are the usual polar coordinates when $d = 2,3$. For example, in $d = 3$,

$$\begin{aligned}
y_1 &= x_1 + r\cos\varphi_1 \cos\theta,\\
y_2 &= x_2 + r\cos\varphi_1 \sin\theta,\\
y_3 &= x_3 + r\sin\varphi_1.
\end{aligned}$$

We want to find the determinant of the Jacobian matrix J. One way to do this is to calculate the metric $g = JJ^T$, which is diagonal with $g_{rr} = \frac{\partial y}{\partial r} \cdot \frac{\partial y}{\partial r} = 1$, $g_{\varphi_j\varphi_j} = \frac{\partial y}{\partial \varphi_j} \cdot \frac{\partial y}{\partial \varphi_j} = r^2 \cos^2\varphi_1 \cdots \cos^2\varphi_{j-1}$, and $g_{\theta\theta} = \frac{\partial y}{\partial \theta} \cdot \frac{\partial y}{\partial \theta} = r^2 \cos^2\varphi \cdots \cos^2\varphi_{d-2}$. We have $\det g = (\det J)^2$, so we take the square root:

$$|\det J| = \sqrt{\det g} = \sqrt{\prod g_{ii}} = r \times (r\cos\varphi_1) \times \cdots \times (r\cos\varphi_1 \cdots \cos\varphi_{d-2}),$$

or $|\det J| = r^{d-1} \cos^{d-2}\varphi_1 \cdots \cos\varphi_{d-2}$.

We don't need to know the sign, but if we want it, we can get it by looking at the point $\varphi_1 = \cdots = \varphi_{d-2} = 0$, $\theta = 0$, $r = 1$, where J is the identity matrix. The sign of the determinant is positive there, and $\{\det J \neq 0\}$ is connected and dense, so the sign is nonnegative everywhere.

The change-of-variables formula tells us that

$$\int_{\mathbb{R}^d} f(y)\,dy = \int_0^\infty \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} f(x+rz)\, r^{d-1} \cos^{d-2}\varphi_1 \cdots \cos\varphi_{d-2}\, d\varphi_1 \cdots d\varphi_{d-2}\, d\theta\, dr.$$

Now we ask the reader to interpret $\int_{|z|=1}$ as shorthand for integration over all the coordinates except r, and dz as an abbreviation for the expression $\cos^{d-2}\varphi_1 \cdots \cos\varphi_{d-2}\, d\varphi_1 \cdots d\varphi_{d-2}\, d\theta$.

Then we do have the identity $\int f(y)\,dy = \int_0^\infty \int_{|z|=1} f(x+rz)\, r^{d-1}\, dz\, dr$, and the derivation in the last section makes sense.

3.3.1. *Exercise: the value of C_d.* This gives us another expression for C_d:

$$C_d = \int_{|z|=1} dz = \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \cos^{d-2}\varphi_1 \cdots \cos\varphi_{d-2}\, d\varphi_1 \cdots d\varphi_{d-2}\, d\theta$$
$$= 2\pi \prod_{j=1}^{d-2} \int_{-\pi/2}^{\pi/2} \cos^j \varphi\, d\varphi.$$

It is fun to use the beta integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and the special value $\Gamma(1/2) = \sqrt{\pi}$ to show that this is equal to $2\pi^{d/2}/\Gamma(d/2)$. Hint: evaluate $\int_{-\pi/2}^{\pi/2} \cos^j \varphi\, d\varphi = \Gamma(\frac{1}{2})\Gamma(j/2+\frac{1}{2})/\Gamma(j/2+1)$.

3.4. **Spherical averages and the distributional Laplacian.** We can get many weighted integrals of the spherical averages $L_f(x;r)$ by evaluating the Laplacian $\nabla^2(f\,d\lambda)$ on certain nonnegative functions.

Lemma 3.7. *Let $x \in \Omega$. Let $R := d(x,\Omega^c)$. Let $\eta \in C_c^\infty(0,R)$ with $\eta \geq 0$. Then there is a nonnegative function $h \in C_c^\infty(\Omega)$ with*

$$\nabla^2(f\,d\lambda)(h) = -\int_0^R \eta'(r) L_f(x;r)\,dr$$

for every locally integrable function f on Ω.

Proof. Let $h(y) := H(|y-x|)$ for $y \in B_R(x)$, where

$$H(r) := \int_r^R \frac{\eta(\rho)}{C_d \rho^{d-1}}\, d\rho.$$

The compactly supported function η is zero on a neighbourhood of 0 and R, so $H(r)$ is smooth, constant near 0, and zero on a neighbourhood of R. Therefore, h is smooth even at x, and compactly supported in $B_R(x)$.

If a smooth function h is radially symmetric around a point x, then there is a formula for its Laplacian which holds in any $\mathbb{R}^d$:

$$\nabla^2 h = \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left[r^{d-1} \frac{\partial}{\partial r} h \right]$$

where we are using the polar coordinates $y = x + rz$.[3]

---

[3]We can get this from the Voss-Weyl formula $\nabla^2 h = \frac{1}{\sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial \xi_i}(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial \xi_j} h)$. Here $g^{ij} = (g^{-1})_{ij}$. Using the coordinates from Section 3.3, we get the result.

The radial derivative of h is of course $\frac{\partial h}{\partial r} = H'(r) = -\eta(r)/C_d r^{d-1}$, which means the Laplacian is $\nabla^2 h = -\eta'(r)/C_d r^{d-1}$. Then

$$\begin{aligned}\nabla^2(f\,d\lambda)(h) &= \int_{B_R(x)} f\,\nabla^2 h\,dy \\ &= -\int_{B_R(x)} f(y)\frac{\eta'(r)}{C_d r^{d-1}}\,dy \\ &= -\int_0^R \int_{|z|=1} f(x+rz)\,\frac{\eta'(r)}{C_d r^{d-1}}\,r^{d-1}\,dz\,dr \\ &= -\int_0^R \frac{\eta'(r)}{C_d}\int_{|z|=1} f(x+rz)\,dz\,dr \\ &= -\int_0^R \eta'(r) L_f(x;r)\,dr.\end{aligned}$$

This is the result. □

We will use this basic result to evaluate the differences $A_f(x;s) - A_f(x;t)$ in terms of the distributional Laplacian. We will get an especially precise result when $\nabla^2(f\,d\lambda) = d\nu$ for some signed measure ν.

3.5. **The difference of averages on two concentric balls: choosing functions for approximation.** Recall that $A_f(x;r)$ is the average of f on the ball $B_x(r)$.

Let $x \in \Omega$, $0 < r < s < d(x,\Omega^c)$. We construct some nonnegative functions η_m that are suitable for Lemma 3.7, and then use it to prove that

$$A_f(x;s) - A_f(x;t) = \lim_{m\to\infty} \nabla^2(f\,d\lambda)(h_m) \tag{2}$$

where $h_m(y) := \int_{|y-x|}^{\infty} \eta_m(r)/C_d r^{d-1}\,dr$ as in the lemma.

Write the formula in Corollary 3.6 as $A_f(x;s) = \int_0^\infty \mathbb{1}_{r<s} \frac{dr^{d-1}}{s^d} L_f(x;r)\,dr$. Then we can write the difference $A_f(x;t) - A_f(x;s)$ as

$$A_f(x;t) - A_f(x;s) = \int_0^\infty \left[\mathbb{1}_{r<t}\frac{dr^{d-1}}{t^d} - \mathbb{1}_{r<s}\frac{dr^{d-1}}{s^d}\right] L_f(x;r)\,dr.$$

We want to connect this to Lemma 3.7. Let w be the function

$$w(r) := \begin{cases} dr^{d-1} & \text{if } r < 1, \text{ and} \\ 0 & \text{if } r \geq 1. \end{cases} \tag{3}$$

Let $W(r) := \min\{1, r^d\}$ be the integral of w from 0 to r. Then the expression in brackets is the derivative of $W(r/t) - W(r/s)$, where it is differentiable. Unfortunately, W isn't smooth, so we can't plug it into Lemma 3.7 directly and we have to approximate.

Let w_m be a sequence of compactly supported, nonnegative smooth functions $w_m \in C_c^\infty(0,\infty)$ which increase to w. Let $W_m(r) := \int_0^r w_m(\rho)\, d\rho$.

Lemma 3.8. *Given $0 < t < s$, the sequence of functions*

$$\eta_{m,s,t}(r) := W_m(r/t) - W_m(r/s)$$

converges uniformly to $W(r/t) - W(r/s)$, and each function satisfies

$$0 \leq \eta_{m,s,t}(r) \leq \begin{cases} \min\{1, r^d/t^d\} & \textit{if } r < s \\ 0 & \textit{otherwise} \end{cases}$$

and

$$|\eta'_{m,s,t}(r)| \leq \begin{cases} d r^{d-1}/t^d & \textit{if } r < s \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Here $\eta_{m,s,t}(r) \geq 0$ because W_m is increasing and $r/t > r/s$, and $\eta_{m,s,t}(r) \leq W_m(r/t) \leq W(r/t) = \min\{1, (r/t)^d\}$. The rest is easy. □

Lemma 3.9. *Suppose $x \in \Omega$ and $0 < t < s$ with $B_s(x) \subseteq \Omega$. If we choose functions $\eta_{m,s,t}$ as above, then we will have*

$$A_f(x;s) - A_f(x;t) = \lim_{m\to\infty} \nabla^2(f\, d\lambda)(h_{m,s,t,x}) \tag{2}$$

for $h_{m,s,t,x}(y) := \int_{|y-x|}^\infty \eta_{m,s,t}(r)/C_d r^{d-1}\, dr$ as in Lemma 3.7.

Proof. We start from Corollary 3.6, and remember that the functions $w_m(r)$ are functions that increase to dr^{d-1} for $r < 1$.

$$\begin{aligned} A_f(x;s) &= \int_0^s \frac{dr^{d-1}}{s^d} L_f(x;r)\, dr \\ &= \int_0^\infty \lim_{m\to\infty} \frac{w_m(r/s)}{s} L_f(x;r)\, dr. \end{aligned}$$

That is bounded by $\mathbb{1}_{r<s}(d/s)|L_f(x;r)|$, which is integrable by Lemma 3.4. So we can use dominated convergence to get:

$$A_f(x;s) = \lim_{m\to\infty} \int_0^\infty \frac{w_m(r/s)}{s} L_f(x;r)\, dr.$$

Replace s by t and subtract:

$$\begin{aligned} A_f(x;s) - A_f(x;t) &= \lim_{m\to\infty} \int_0^\infty \left[\frac{w_m(r/s)}{s} - \frac{w_m(r/t)}{t} \right] L_f(x;r)\, dr \\ &= \lim_{m\to\infty} \int_0^\infty (-\eta'_{m,s,t}(r)) L_f(x;r)\, dr \\ &= \lim_{m\to\infty} \nabla^2(f\, d\lambda)(h_{m,s,t}). \end{aligned}$$

The last equality comes from Lemma 3.7. This is the desired identity. □

If $\nabla^2(f\,d\lambda)$ is a signed measure $d\nu$, we can use this lemma to write the difference $A_f(x;t) - A_f(x;s)$ as an integral with respect to the signed measure.

3.6. **The difference of averages on two concentric balls: a formula for signed measures.**

Theorem 3.10. *Suppose f is locally integrable on Ω and $\nabla^2(f\,d\lambda) = d\nu$ where ν is a signed measure. Let $x \in \Omega$ and $0 < t < s < R = d(x, \Omega^c)$.*

Let

$$h_{s,t,x}(y) := \int_{|y-x|}^{\infty} \frac{W(r/t) - W(r/s)}{C_d r^{d-1}}\,dr.$$

Then $\int h_{s,t,x}\,d\nu = A_f(x;s) - A_f(x;t)$.

Proof. Let $h_{m,s,t,x}$ be the functions provided by Lemma 3.9 with

$$A_f(x;s) - A_f(x;t) = \lim_{m\to\infty} \nabla^2(f\,d\lambda)(h_{m,s,t,x}).$$

If we can prove that $\lim_m \int h_{m,s,t,x}\,d\nu = \int h_{s,t,x}\,d\nu$, we will be done. We can write the difference as

$$h_{s,t,x}(y) - h_{m,s,t,x}(y) = \int_{|y-x|}^{\infty} \frac{W(r/t) - W(r/s) - \eta_{m,s,t}(r)}{C_d r^{d-1}}\,dr.$$

This difference is uniformly bounded in absolute value by

$$\int_0^{\infty} \frac{|W(r/t) - W(r/s) - \eta_{m,s,t}(r)|}{C_d r^{d-1}}\,dr.$$

Lemma 3.8 says that the integrand is uniformly bounded by $\mathbb{1}_{r<s} r C_d / t^d$, and that it goes to zero pointwise. By the dominated convergence theorem, $\max |h_{s,t,x} - h_{m,s,t,x}| \to 0$, so we do have $\lim_m \int h_{m,s,t,x}\,d\nu = \int h_{s,t,x}\,d\nu$. ☐

This lemma will allow us to get an estimate on the difference of averages from weak estimates on the Laplacian. To get it, we need to know $\int h\,dx$.

Lemma 3.11. *Let $x \in \Omega$ and $0 < t < s$. With $h_{s,t,x}$ defined as in Theorem 3.10,*

$$\int h_{s,t,x}(y)\,dy = \frac{1}{2(d+2)}(s^2 - t^2).$$

Proof. We know what the function is, so the proof is a calculation. First,

$$\begin{aligned}\int_{\mathbb{R}^d} h_{s,t,x}(y)\,dy &= \int_{\mathbb{R}^d}\left[\int_{|x-y|}^{s}\frac{W(\rho/t)-W(\rho/s)}{C_d\rho^{d-1}}\,d\rho\right]dy\\ &= \int_0^\infty\left[\int_r^s\frac{W(\rho/t)-W(\rho/s)}{C_d\rho^{d-1}}\,d\rho\right]C_d r^{d-1}\,dr\\ &= \int_0^s\frac{W(\rho/t)-W(\rho/s)}{\rho^{d-1}}\left[\int_0^\rho r^{d-1}\,dr\right]d\rho\\ &= \int_0^s(W(\rho/t)-W(\rho/s))\frac{\rho}{d}\,d\rho.\end{aligned}$$

We have $W(r)=\min\{1,r^d\}$, so

$$\begin{aligned}\int_0^s(W(\rho/t)-W(\rho/s))\frac{\rho}{d}\,d\rho &= \left[\int_0^t\frac{\rho^{d+1}}{dt^d}\,d\rho+\int_t^s\frac{\rho}{d}\,d\rho\right]-\int_0^s\frac{\rho^{d+1}}{ds^d}\,d\rho\\ &= \frac{t^2}{d(d+2)}+\frac{s^2-t^2}{2d}-\frac{s^2}{d(d+2)}\\ &= \frac{1}{2(d+2)}(s^2-t^2).\end{aligned}$$

□

We will use this in Lemma 3.15 and Theorem 3.16 to get quadratic bounds on $A_f(x;s)-A_f(x;t)$ in the case where $\nabla^2(f\,d\lambda)=\rho\,d\lambda$ with $0\le\rho\le 1$.

3.7. **Limits of radial averages.** A function is a *limit of radial averages at* x if it is integrable in a neighbourhood of x and the limit $\lim_{r\to 0}A_f(x;r)$ exists and is equal to $f(x)$. This is strictly weaker than continuity at a point.

A subharmonic function is always a limit of radial averages, because

$$h(x)\le\inf_{0<r<s}A_h(x;r)\le\limsup_{y\to x}h(y)\le h(x).$$

3.7.1. *Subharmonicity on average.* We say that a function f is *subharmonic on average* at x if it is a limit of radial averages at x and satisfies condition (b) in the definition of subharmonicity. That is, there exists some small $\varepsilon>0$ so that

$$\lim_{r\to 0}A_f(x;r)=f(x)=\inf_{0<r<\varepsilon}A_f(x;r).$$

This is strictly weaker than subharmonicity. For example, the sign function is subharmonic on average everywhere, but its average on the interval $B_2(1)=(-1,3)$ is $A_f(1;2)=\frac{1}{2}$, which is strictly less than $\operatorname{sign}1=1$.

From our point of view, the problem is that the distributional Laplacian of the sign function is too irregular: it is not a signed measure.[4]

3.7.2. *...implies positivity of the Laplacian.* We will show that, if the Laplacian is a signed measure $d\nu$, then ν is positive on any measurable set where f is subharmonic on average.

We need a consequence of the Lebesgue-Besicovitch theorem.

Lemma 3.12. *If $E \subseteq \Omega$ is measurable and ν is a signed measure with $\nu(E) < 0$, then there is a point $x \in E$ with*

$$\limsup_{t\to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} < 0.$$

Proof. Let $\mu = |\nu| + \lambda$, and $f = d\nu/d\mu$. Then $\int f\, d\mu = \nu(E) < 0$, so the set of points where f is negative must have positive measure.

By the Lebesgue-Besicovitch differentiation theorem,

$$\lim_{t\to 0} \frac{\nu(B_t(x))}{\mu(B_t(x))} = f(x)$$

except on a set N with $\mu(N) = 0$. The set of points with $f(x) < 0$ has positive measure, so there must be some point x with $f(x) < 0$ and $x \notin N$.

By definition of the limit, $\nu(B_t(x))$ is negative for small t, and $0 \leq \lambda \leq \mu$, so $1/\lambda(B_t(x)) \geq 1/\mu(B_t(x))$ and $\nu(B_t(x))/\lambda(B_t(x)) \leq \nu(B_t(x))/\mu(B_t(x))$.

We therefore have the strict inequality

$$\limsup_{t\to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} \leq \limsup_{t\to 0} \frac{\nu(B_t(x))}{\mu(B_t(x))} = f(x) < 0.$$

That proves the result. □

Theorem 3.13. *Suppose f is locally integrable on Ω, and $\nabla^2(f\, d\lambda) = d\nu$ where ν is a signed measure.*

If f is subharmonic on average on a measurable set E, then E is a positive set for ν, i.e. $\nu(E') \geq 0$ for $E' \subseteq E$.

Proof. Suppose E is not positive. Let E' be a measurable subset of E with negative ν-measure. By Lemma 3.12, $\exists x \in E'$ with

$$\limsup_{t\to 0} \frac{\nu(B_t(x))}{\lambda(B_t(x))} = -c < 0.$$

Let $s > 0$ be small enough that $\nu(B_t(x))/\lambda(B_t(x)) < -c/2$ for $t < s$ and the subharmonic inequality holds for $B_s(x)$.

---

[4]If h is a test function, then $\nabla^2(\mathrm{sign})(h) = \int_{-\infty}^{\infty} \mathrm{sign}\, x\, h''\, dx = -2h'(0)$. Suppose there were a signed measure ν with $\int h\, d\nu = -2h'(0)$. Then there would be a constant $C = |\nu|(\Omega)/2$ with $|h'(0)| \leq C \max |h|$ for every test function h, but this is absurd.

Theorem 3.10 and Corollary 3.11 tell us that, for each $x \in \Omega$ and $0 < t < s$, there is a nonnegative radially symmetric continuous function $h_{s,t,x}$ with

$$\int h_{s,t,x}(y)\, d\nu(y) = A_f(x;s) - A_f(x;t).$$

The reader can check from the definition that $h_{s,t,x}(y)$ is supported on B_s, radially symmetric, and decreases as y gets farther from x. So if $\alpha > 0$, then $\{h_{s,t,x} > \alpha\}$ is a ball around x of radius less than s, and

$$\begin{aligned}\int h_{s,t,x}\, d\nu &= \int_0^\infty \nu(\{h_{s,t,x} > \alpha\})\, d\alpha \\ &\le -\frac{c}{2}\int_0^\infty \lambda(\{h_{s,t,x} > \alpha\})\, d\alpha \\ &= -\frac{c}{2}\int h_{s,t,x}\, d\lambda = -\frac{c}{4(d+2)}(s^2 - t^2).\end{aligned}$$

The last step is Lemma 3.11.

Fix s, take $t \to 0$, and use the fact that $A_f(x;t) \to f(x)$ as $t \to 0$, because f is a limit of radial averages at x. We get the impossible inequality:

$$0 \le A_f(x;s) - f(x) = \limsup_{t \to 0} \int h_{s,t,x}\, d\nu \le -\frac{c}{4(d+2)} s^2 < 0.$$

So, there is no measurable subset E' with $\nu(E') < 0$. □

3.8. **On an open set, positivity implies subharmonicity.** Now we will go from the Laplacian to full subharmonicity.

Lemma 3.14. *Let f be locally integrable on an open set Ω. Suppose $\nabla^2(f\, d\lambda)$ is positive on Ω. Then there is a subharmonic $\bar{f}$ on Ω with $\bar{f} = f$ a.e. on Ω.*

Proof. Let $x \in \Omega$ and $t < s < d(x, \Omega^c)$.

Theorem 3.10 tells us that $A_f(x;s) - A_f(x;t) = \nabla^2(f\, dx)(h_{s,t}) \ge 0$ for a certain function $h_{s,t}$, so $A_f(x;t)$ decreases to a limit (possibly $-\infty$) as $t \to 0$.

Let $\bar{f}(x)$ be that limit:

$$\bar{f}(x) := \lim_{t \to 0} A_f(x;t) = \inf_{t > 0} A_f(x;t).$$

The Lebesgue differentiation theorem tells us that $A_f(x;t) \to f(x)$ for almost every x, so $f = \bar{f}$ for almost every $x \in \Omega$.

We claim $\bar{f}$ is subharmonic on Ω. It is less than or equal to its averages on balls because $\bar{f}(x) \le A_f(x;t) = A_{\bar{f}}(x;t)$, so (b) in the definition of subharmonicity is satisfied. We must prove that $\bar{f}$ is upper semicontinuous.

Let x_n be a sequence of points in Ω that converge to x. Let $0 < r < d(x,\Omega^c)$. We can break down $\bar{f}(x_n)$ in the following way:

$$\bar{f}(x_n) = \Big[\bar{f}(x_n) - A_f(x_n;r)\Big] + \Big[A_f(x_n;r) - A_f(x;r)\Big] + A_f(x;r).$$

The first summand is nonpositive by definition. The second one converges to 0 as $n \to \infty$, because $A_f(x;r)$ is continuous in x. This is a general property of convolutions, but it can be proven directly in this case by writing

$$A_f(r;x_n) - A_f(r;x) = \frac{1}{\lambda(B_r)} \int (\mathbb{1}_{B_r(x_n)} - \mathbb{1}_{B_r(x)}) f(y)\, dy.$$

The integrand is dominated by $|f|$ and converges pointwise to zero. Take the lim sup of both sides of the equality as $n \to \infty$:

$$\limsup \bar{f}(x_n) = (\text{nonpositive}) + A_f(x;r)$$

and then take $r \to 0$ to get $\limsup \bar{f}(x_n) \le \bar{f}(x)$. So $\bar{f}$ is upper semicontinuous, and therefore subharmonic. □

Note. Theorems 3.13 and 3.14 together tell us that if f is subharmonic on average, and its Laplacian is a signed measure, then it is subharmonic.

3.9. **The measure of the Laplacian on the zero set.** In what follows, we suppose f is a limit of radial averages, $f \ge 0$, and the distributional Laplacian $\nabla^2(f\, d\lambda)$ is $\rho\, d\lambda$ with $|\rho| \le C$.

We will show that $\rho = 0$ a.e. on the zero set $\{x : f(x) = 0\}$. First, we show that $f(y)$ converges uniformly to zero as y approaches the zero set.

Lemma 3.15. *Suppose $f \ge 0$ is a limit of radial averages and $\nabla^2(f\, d\lambda) = \rho\, d\lambda$ with $|\rho| \le C$. If $f(x) = 0$, then $f(y) \le 2^d C|y-x|^2$ if $|y-x| < \frac{1}{2} d(x,\Omega^c)$.*

By Theorem 3.10, if x is a point in Ω and $0 < t < s < d(x,\Omega^c)$, then

$$A_f(x;s) - A_f(x;t) = \int h_{s,t,x}\, \rho\, d\lambda$$

where $h \ge 0$ and $\int h\, d\lambda = (s^2 - t^2)/2(d+2)$. Therefore,

$$|A_f(x;s) - f(x)| = \lim_{t \to 0} |A_f(x;s) - A_f(x;t)| \le \frac{Cs^2}{2(d+2)}.$$

Fix $x, y \in \Omega$ with $|y-x| < \frac{1}{2} d(x,\Omega^c)$. Suppose $f(x) = 0$. Write

$$\begin{aligned} f(y) &= f(y) - f(x) \\ &= [f(y) - A_f(y;s)] + [A_f(y;s) - 2^d A_f(x;2s)] + 2^d [A_f(x;2s) - f(x)]. \end{aligned}$$

The first summand is bounded in absolute value by $Cs^2/2(d+2)$, the second one is nonpositive because $B_s(y) \subseteq B_{2s}(y)$ and $\lambda(B_{2s}) = 2^d\lambda(B_s)$, and the third one is bounded by $2^{d+2}Cs^2/2(d+2)$. So,

$$f(y) \leq Cs^2 \frac{1+2^{d+2}}{2(d+2)} \leq 2^d Cs^2.$$

This is the result. □

Theorem 3.16. *Suppose $f \geq 0$ is a limit of radial averages and $\nabla^2(f\,d\lambda) = \rho\,d\lambda$ with $|\rho| \leq C$. Then $\rho = 0$ a.e. on the zero set $\{x \in \Omega : f(x) = 0\}$.*

Proof. Let $Z := \{x \in \Omega : f(x) = 0\}$. By the Lebesgue density theorem, there is a set of zero λ-measure N so that for every point $x \in Z \setminus N$, both of the following equalities hold:

$$\lim_{r\to 0} \frac{\lambda(B_r(x) \cap Z^c)}{\lambda(B_r)} = 0 \qquad \text{and} \tag{4}$$

$$\lim_{r\to 0} \frac{1}{\lambda(B_r)} \int_{B_r(x)} \rho\,d\lambda = \rho(x). \tag{5}$$

By the last theorem, $f(y) \leq 2^d C|y-x|^2$ for y sufficiently close to x, so

$$A_f(x;s) \leq \frac{\lambda(B_s(x) \cap Z^c)}{\lambda(B_s(x))} \times O(s^2) = o(s^2).$$

This estimate holds for every $x \in Z \setminus N$.

For every $x \in Z$ and $s < d(x, \Omega^c)$, we have the inequality $f(x) = 0 \leq A_f(x;s)$, so f is subharmonic on average on Z. The Laplacian is the signed measure $\rho\,d\lambda$, and by Theorem 3.13, that signed measure must be positive on Z, which means that we must have $\rho \geq 0$ almost everywhere on Z.

So it is enough to prove $\rho \leq 0$ a.e. on the zero set. Suppose not. Then there must be at least one point $x \in Z \setminus N$ with $\rho(x) > 0$. The point is not in N, so the limit in equation 5 exists. For $s > 0$ sufficiently small, we have

$$\inf_{t\in(0,s)} \frac{1}{\lambda(B_t)} \int_{B_t(x)} \rho\,d\lambda > \frac{\rho(x)}{2} \qquad \text{for } t \in (0,s).$$

Construct $h_{s,t,x}$ as in Theorem 3.10, and repeat the reasoning in the proof of Lemma 3.13 to get the inequality

$$\int h_{s,t,x}\,\rho\,d\lambda \geq \frac{\rho(x)}{2} \int h_{s,t,x}\,d\lambda.$$

Then by Theorem 3.10 and the Lemma 3.11,

$$A_f(x;s) - A_f(x;t) = \int h_{s,t}\,\rho\,d\lambda \geq \frac{\rho(x)}{2} \int h_{s,t}\,d\lambda = \frac{\rho(x)}{4(d+2)}(s^2 - t^2).$$

Take the limit of both sides as $t \to 0$ and use the fact that $\lim_{t\to 0} A_f(x;t) = f(x) = 0$ to get the inequality

$$A_f(x;s) \geq \frac{\rho(x)}{4(d+2)} s^2.$$

This contradicts the estimate $A_f(x;s) = o(s^2)$ for $x \in Z$.

Therefore, $\rho = 0$ a.e. on the zero set Z. □

4. THE EXISTENCE OF QUADRATURE DOMAINS

We say that a weight function is *properly supported* if it is greater than or equal to 1 on some bounded open set, and 0 outside that open set. For example, any finite sum of indicator functions of bounded open sets is properly supported, but the function $\frac{1}{2}\mathbb{1}_{B_1}$ is not properly supported.

From now on, suppose w is a properly supported weight function. We will prove the existence of a quadrature domain for w. We start by posing a minimization problem, then extract a set from the solution, and finally prove that the set is a quadrature domain in Theorem 4.16.

Definition. If ψ and ψ' are distributions, then we say that $\psi \leq \psi'$ if $\psi' - \psi$ is a positive distribution.

Let $RA(\mathbb{R}^d)$ be the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ which are a limit of radial averages at every point in $\mathbb{R}^d$. This is a weak continuity condition.

The minimization problem is:

Minimization problem.
Find the smallest nonnegative $f \in RA(\mathbb{R}^d)$ with $\nabla^2(f\,d\lambda) \leq (1-w)\,d\lambda$.

It will turn out that there is a function that is pointwise less than or equal to any other function, and that will be the "smallest" one.

Some sort of weak continuity condition is necessary in this problem. If we allow the whole class of nonnegative locally integrable functions, no function can be minimal except for 0, which is typically not a solution.

4.1. Newtonian potentials.

If w is a bounded, compactly supported weight function, let the *Newtonian potential* of w be the convolution of w with the unrestricted Green's function:

$$Nw(x) := \int_{\mathbb{R}^n} G(x,y) w(y)\, dy.$$

Recall that $G(x,y)$ is a function of $x-y$ only, so this is really a convolution.

The convolution of a bounded, compactly supported function with a locally integrable function is continuous. Green's function is locally integrable, like we said earlier, and the first derivatives $\partial G/\partial x_i = -x_i/C_d|x-y|^d$ are also locally integrable.

Therefore, Nw is continuous on all of $\mathbb{R}^d$. We claim that it has continuous first derivatives which are $\int_{\mathbb{R}^n}(-x_i/C_d|x-y|^d)w(x)\,dx$. This is easy to check by integrating the functions using Fubini's theorem and checking that the result is Nw plus a function independent of x_i.

Green's function is symmetric in its variables, so if $h \in C_c^\infty(\mathbb{R}^d)$ is a test function, then $-\nabla^2(Nw\,d\lambda)[h] = (Nw\,d\lambda)[-\nabla^2 h] = (w\,d\lambda)[-N\nabla^2 h] = w\,d\lambda(h)$ by Lemma 2.1. Therefore, $-\nabla^2(Nw\,d\lambda) = w\,d\lambda$ as distributions.

If w is bounded and *constant* outside a compact set, let c be the constant, and define $Nw := N(w-c) + \frac{c}{2d}|x|^2$. Again $-\nabla^2(Nw\,d\lambda) = w$.

4.2. **Minimization over superharmonic functions.** We use Newtonian potentials to transfer the minimization problem to the theory of superharmonic functions.

Definitions. If f is a function, it's *superharmonic* if $-f$ is subharmonic. If ψ is a distribution on E, then it is *negative* if $\psi[h] \leq 0$ for $h \in C_c^\infty(E)$, $h \geq 0$.

The theorems about subharmonic functions carry over to superharmonic functions with a minus sign. In particular, we have these three statements:

Theorem 4.1. *If f is a superharmonic function on an open set Ω, then $\nabla^2(f\,d\lambda)$ is a negative distribution.*

Theorem 4.2. *If $\nabla^2(f\,d\lambda) = d\nu$, and f is superharmonic on average on a measurable set E, then E is a negative set for ν.*

Lemma 4.3. *If the Laplacian of f is a negative distribution on an open set, then f is equal almost everywhere to a superharmonic function.*

This is enough to turn the minimization problem into a question about superharmonic functions.

Theorem 4.4. *The minimization problem is equivalent to:*

> *Find the smallest nonnegative function f on $\mathbb{R}^d$ with the property that the sum $f + N(1-w)$ is superharmonic everywhere in $\mathbb{R}^d$.*

Proof. We will show that the two classes of functions are the same: f is a limit of radial averages with $\nabla^2(f\,d\lambda) \leq (1-w)\,d\lambda$ if and only if $f + N(1-w)$ is superharmonic.

Suppose f is a limit of radial averages and $\nabla^2(f\,d\lambda) \leq (1-w)\,d\lambda$. Then

$$\nabla^2[(f + N(1-w))\,d\lambda] = \nabla^2(f\,d\lambda) - (1+w)\,d\lambda \leq 0.$$

By Lemma -3.14, $f + N(1-w) = \bar{f}$ a.e. for some superharmonic $\bar{f}$.

The function $N(1-w)$ is continuous and averaging is linear, so $f + N(1-w)$ is a limit of radial averages, and so is the superharmonic function $\bar{f}$. Therefore, they are equal, and $f + N(1-w) = \bar{f}$ is superharmonic.

Suppose $f + N(1-w)$ is superharmonic. By Theorem −3.1,

$$\nabla^2\big[(f + N(1-w))\,d\lambda\big] \le 0,$$

so $\nabla^2(f\,d\lambda) \le (1-w)\,d\lambda$.

Every superharmonic function is a limit of radial averages, and $N(1-w)$ is continuous, so by linearity f is a limit of radial averages.

The two conditions are therefore equivalent. □

We can now use the fundamental convergence theorem for superharmonic functions to find a minimum.

Theorem 4.5 (Fundamental convergence theorem)**.**
Let Γ be a family of superharmonic functions defined on an open subset of $\mathbb{R}^d$ and locally uniformly bounded below. Let $u(x)$ be the pointwise infimum of all the functions in Γ. Let $u_+(x) = \min\{u(x), \liminf_{y\to x} u(y)\}$.
Then $u_+ = u$ almost everywhere, and u_+ is superharmonic.

Proof. See for example Section 1.III.3 of Doob [1].

Corollary 4.6. *Let π be a measurable, bounded function on $\mathbb{R}^d$ that is constant outside a compact set. Then there is a smallest $f \ge 0$ with the property that $f + N\pi$ is superharmonic.*

Proof. Let $\Gamma = \{u : u \text{ is superharmonic}, \gamma \ge N\pi\}$. The functions in this class are uniformly bounded below on any compact set K by $\min_K N\pi$.

Apply the fundamental convergence theorem to Γ to get a superharmonic function u_+ less than or equal to every function in Γ. Then $u_+ \in \Gamma$:

$$\begin{aligned} u_+(x) &= \min\{u(x), \liminf_{y\to x} u(y)\} \\ &\ge \min\{N\pi(x), \liminf_{y\to x} N\pi(y)\} \\ &\ge N\pi(x) \end{aligned}$$

by continuity of $N\pi(x)$, so it obeys the inequality, and it's superharmonic.

Set $f := u_+ - N\pi$. Then $f \ge 0$ and $f + N\pi$ is superharmonic, and if that is true for g, then $g + N\pi \in \Gamma$ and $f + N\pi = u_+ \le g + N\pi$, so $f \le g$. □

Corollary 4.7. *There is a smallest function $f \ge 0$ that is a limit of radial averages and satisfies $\nabla^2(f\,d\lambda) \le (1-w)\,d\lambda$.*

Proof. Combine Corollary 4.6 with Theorem 4.4. □

In the next section, we will characterize the Laplacian of the minimal function, and discover that there is a quadrature domain hiding inside it.

4.3. **Finding the Laplacian.** Suppose w is a properly supported weight function. Let $f \geq 0$ be the minimal limit of radial averages with $\nabla^2(f\,d\lambda) \leq (1-w)\,d\lambda$ promised by Corollary 4.7.

By Corollary 3.2,

$$\nabla^2(f\,d\lambda) - (1-w)\,d\lambda = -d\mu$$

for some locally finite measure μ.

The function $f + N(1-w)$ is superharmonic, and therefore lower semicontinuous. So, f is also lower semicontinuous, and $A := \{f > 0\}$ is open.

Lemma 4.8. $\mu(A) = 0$.

Proof. Let $\gamma = f + N(1-w)$ as before. Then $\nabla^2(\gamma d\lambda) = -d\mu$.

Let $x \in A$, so $\gamma(x) > N[1-w](x)$. The left-hand function in that inequality is lower semicontinuous, and the right-hand one is continuous, so it is possible to choose a small radius $r > 0$

$$\min_{y \in B_r(x)} \gamma(y) > \max_{y \in B_r(x)} N[1-w](y).$$

If we have a superharmonic function γ on Ω, we can replace its values on a ball $B, \overline{B} \subseteq \Omega$ with a harmonic function that has the same boundary values while preserving the function outside of that ball, and the resulting γ' will still be superharmonic and $\gamma' \leq \gamma$. See e.g. I.II.6 of Doob [1].

We carry out this operation on γ using the ball $B_r(x)$, and we get a new function γ'. This function still satisfies the inequality $\gamma' \geq N(1-w)$, because

$$\gamma'(y) \geq \min_{y \in B_r(x)} \gamma(y) > N[1-w](y) \qquad \forall y \in B_r(x)$$

for y in the ball $B_r(x)$, and $\gamma' = \gamma \geq N(1-w)$ outside of the ball. But γ is the minimal superharmonic function with $\gamma \geq N(1-w)$, so $\gamma = \gamma'$. That means in particular that γ is harmonic inside the ball $B_r(x)$.

If a function is harmonic on an open set, then its Laplacian is zero on that set. So, $\nabla^2(\gamma d\lambda)[h] = -\int h\,d\mu = 0$ for any function $h \in C_c^\infty(B_r(x))$, and that means that $\mu(B_r(x)) = 0$. That's true for some ball around any point x, and we can cover A by countably many such balls, so $\mu(A) = 0$. □

We can use the Lebesgue density theorem to get rough bounds on ∂A.

Theorem 4.9. *If* $E \subseteq A^c$, *then* $0 \leq \mu(E) \leq \lambda(E)$.

Proof. We have $\nabla^2 f \leq (1-w)\,d\lambda$, so $d\mu \geq 0$. That's the lower bound.

By definition, $f + N(1-w)$ is superharmonic, and $N(1-w)$ is continuous, so f is a limit of radial averages. If x is any point in ∂A, then

$$f(x) = 0 \leq \frac{1}{\lambda(B_r)} \int_{B_r(x)} f(y)\,dy.$$

Therefore, f is subharmonic on average on ∂A. Use Lemma 3.13 to get $\nabla^2(f\,d\lambda) \geq 0$ on ∂A. Then, $d\mu = (1-w)\,d\lambda - \nabla^2(f\,d\lambda) \leq d\lambda$. □

Corollary 4.10. $\nabla^2(f\,dx) = (1-w)\mathbb{1}_A\,d\lambda$.

By the theorem, the measure μ is absolutely continuous with respect to λ, so there is a Radon-Nikodym derivative $d\mu/d\lambda$ and $0 \leq d\mu/d\lambda \leq 1$ a.e. We choose a version with $0 \leq d\mu/d\lambda \leq 1$ everywhere.

Let ρ be the difference $1-w-d\mu/d\lambda$. Then

$$\rho\,d\lambda = (1-w)\,d\lambda - d\mu = \nabla^2(f\,d\lambda).$$

We will be done if we can show that $\rho = (1-w)\mathbb{1}_A$ almost everywhere.

By Theorem 4.8, $\mu(A) = 0$, so $\rho = 1-w$ almost everywhere on A. On the other hand, we are in precisely the setting of Theorem 3.16, so we have $\rho = 0$ almost everywhere on $\{f=0\} = A^c$. Therefore $\rho = (1-w)\mathbb{1}_A$ a.e. □

4.4. **A quadrature domain for Green's functions.** We are now able to exhibit the quadrature domain, although we're only about three-quarters of the way to the proof that it really is one.

In the last section, we showed that, if f is a solution of the minimization problem, then $\nabla^2(f\,d\lambda) = (1-w)\mathbb{1}_A\,d\lambda$ where $A = \{f > 0\}$. The lemma below uses that to get a formula for f in terms of A.

Theorem 4.11. *If f solves the minimization problem, then $f = N\big[(w-1)\mathbb{1}_A\big]$.*

Proof. Let $\varphi := (w-1)\mathbb{1}_A$. As in Section 4.1, $-\nabla^2(N\varphi\,d\lambda) = \varphi\,d\lambda$, so the distributional Laplacian of $(f-N\varphi)\,d\lambda$ is zero on $\mathbb{R}^d$. By Lemma 3.14, $f-N\varphi$ is subharmonic and superharmonic, so it is harmonic.

We want to show that it's zero. Our strategy will be to prove that it goes to zero at infinity and use Liouville's theorem.

We'll prove below in Corollary 4.14 that A is always bounded. If $d \geq 3$, then the unrestricted Green's function is $|x-y|^{2-d}/(d-2)C_d$, which goes to zero uniformly for $y \in A$ as $x \to \infty$. Therefore,

$$N\varphi(x) = \int G(x,y)\varphi(y)\,dy \to 0$$

and we can apply Liouville's theorem to conclude that $f = N\varphi$.

What about when the dimension is smaller? In $d = 2$, we'll use a similar strategy, but we need the fact that $\int \varphi\,dy = 0$. Let h be a smooth, compactly supported function which is 1 on A. Then

$$\int h\varphi\,d\lambda = (\varphi\,d\lambda)[h] = (f\,d\lambda)[\nabla^2 h] = \int f\nabla^2 h\,d\lambda = 0,$$

because $\nabla^2 h = 0$ on A and $f = 0$ outside A. But $\varphi = h\varphi$, so $\int \varphi \, d\lambda = 0$. So $\int G(x,0)\varphi(y)\, dy = 0$, and we again have

$$\begin{aligned} N\varphi(x) &= \int G(x,y)\varphi(y)\, dy \\ &= \int (G(x,y) - G(x,0))\varphi(y)\, dy \\ &= \int \frac{1}{\pi} \log \frac{|x|}{|x-y|} \varphi(y)\, dy \to 0. \end{aligned}$$

In one dimension, we also want to have $\int y\varphi \, dy = 0$. Prove that the same way using $h \in C_c^\infty$ with $h(y) = y$ for $y \in A$. Then $G(x,y) = -\frac{1}{2}|x-y|$, so

$$\int G(x,y)\varphi(y)\, dy = \pm \frac{1}{2} \int (y-x)\varphi(y)\, dy$$

for $x \geq \max A$ and $x \leq \min A$. This is zero because the first and second moments are zero. In each case, we conclude that $f = N\varphi$. □

Corollary 4.12. *Let $Q := A \cup \{w \geq 1\}$. Then $N[w - \mathbb{1}_Q]$ is nonnegative everywhere and zero outside Q.*

Proof. We chose our function f with $\nabla^2(f\, d\lambda) \leq (1-w)\, d\lambda$, and Corollary 4.10 tells us that $\nabla^2(f\, d\lambda) = (1-w)\mathbb{1}_A \, d\lambda$. Therefore, $(1-w)\mathbb{1}_A \leq 1-w$ almost everywhere, or in other words $0 \leq (1-w)\mathbb{1}_{A^c}$ almost everywhere.

Part of the definition of a good weight function is that $w = 0$ or $w \geq 1$ at every point, so $w \in \{0,1\}$ almost everywhere on A^c. Therefore, on that set, $\mathbb{1}_Q = \mathbb{1}_{\{w-1\}} = w$ almost everywhere, so $N[w - \mathbb{1}_Q] = N[(w-1)\mathbb{1}_A] = f$. And f is nonnegative everywhere and zero outside Q. □

This set Q is our quadrature domain, and we'll prove that in the next section.

4.4.1. *A is bounded.* We will prove that A is bounded by exhibiting a function f which is compactly supported and satisfies the conditions of the minimization problem. First we compute the Newtonian potential of the ball.

Lemma 4.13.

$$N\mathbb{1}_{B_r} = \begin{cases} c_1 - |x|^2/2d & |x| \leq r \\ G(x,0)\lambda(B_r) & |x| \geq r \end{cases},$$

where c_1 is a constant chosen to make the function continuous.

Proof. If $|x| \geq r$, then the function $y \mapsto G(x,y)$ is harmonic on B_r, so $\int G(x,y)\mathbb{1}_{B_r}\, dy$ is equal to $G(x,0)$ times the measure of B_r.

Let $h(x) := N\mathbb{1}_{B_r} + |x|^2/2d$. This function is continuous. The distributional Laplacian of h is zero inside the ball, because $\nabla^2 N\mathbb{1}_{B_r} = -\mathbb{1}_{B_r}$ and

$\nabla^2|x|^2 = 2d$. By Lemma 3.14 applied to both h and $-h$, it is both subharmonic and superharmonic, so h is harmonic inside the ball.

The Newtonian potential of the ball is radially symmetric, and so is $|x|^2/2d$, so the value of h on the sphere $|x| = r$ is some constant. By the maximum principle, h is constant inside the ball, so $N\mathbb{1}_{B_r} = c_1 - |x|^2/2d$. □

Corollary 4.14. *Let $c = 1 \vee \max w$. Let the support of w be contained in B_R. Then $A \subseteq B_{c^{1/d}R}$.*

Proof. Let h be $N[c\mathbb{1}_{B_R}] - N[\mathbb{1}_{B_{c^{1/d}R}}]$. This is continuous, and $h + N(1-w)$ is the Newtonian potential of a nonnegative function and therefore superharmonic. The lemma tells us that

$$h = \begin{cases} c_1 - (c-1)|x|^2/2d & |x| \le R \\ c_2 + cG(x,0)\lambda(B_R) + |x|^2/2d & R \le |x| \le c^{1/d}R \\ 0 & |x| \ge c^{1/d}R. \end{cases}$$

Here c_1, c_2 are determined by continuity. The radial derivative of h is

$$\frac{\partial h}{\partial r} = \begin{cases} -(c-1)r/d & r \le R \\ r(1 - cR^d/r^d)/d & R \le r \le c^{1/d}R \\ 0 & r \ge c^{1/d}R \end{cases}$$

where $r = |x|$. This is always less than or equal to zero, and h is zero for large x, so h is nonnegative. By the choice of f, $f \le h$, so $A \subseteq B_{c^{1/d}R}$. □

4.5. **Extending to all integrable subharmonic functions.** Here is the grand climax of this chapter, Sakai's Lemma 5.1 [6]. For completeness, we present the proof and various minor results that are used.

Theorem 4.15 (Sakai's Lemma 5.1)**.** *Let Q be an open bounded set. If there is a function $\varphi \in L^\infty(Q)$ with $N\varphi \ge 0$ on Q and $N\varphi = 0$ on Q^c, then*

$$\int s\varphi \, dy \ge 0$$

for every superharmonic function s on Q with $\int s\mathbb{1}_{\{s>0\}}\, dx < \infty$.

Note. This says in particular that, if a linear functional φ on $L^1(Q)$ is nonnegative on G_x for $x \in \mathbb{R}^d$ and $-G_x$ for $x \in Q^c$, then it's nonnegative on all superharmonic functions in $L^1(Q)$.

Proof. The basic idea is approximation, but it's delicate and relies on a tight estimate of the modulus of continuity of $N\varphi$. Without loss of generality, we can assume that $|\varphi| \le 1$ everywhere.

Let s be an integrable superharmonic function on Q. Let $s_n := s * \psi_n$ be the approximations defined in Theorem 3.1. As before, each function s_n is defined on $\{d(x, \Omega^c) > 1/n\}$, and it's smooth and superharmonic.

Let $h_j \in C_c^\infty(Q)$ be test functions $0 \le h_j \le 1$ converging pointwise to 1. We will choose them carefully later. The function $s \vee 0$ is integrable by assumption. Use Fatou's lemma on the nonnegative functions $(s \vee 0) - sh_j\varphi$ and their limit $(s \vee 0) - s\varphi$ to get the inequality

$$\limsup_{j\to\infty} \int sh_j\varphi\, dy \le \int s\varphi\, dy.$$

Each h_j is supported on a compact subset of Q, so we can introduce the approximations s_n, and we get

$$\limsup_{j\to\infty} \lim_{n\to\infty} \int s_n h_j \varphi\, dy \le \int s\varphi\, dy.$$

Here $s_n h_j$ are smooth and compactly supported in Q when n is large enough for a fixed j. By Lemma 2.1 and the fact that Green's function is symmetric,

$$\int s_n h_j \varphi\, dy = \int N(-\nabla^2)(s_n h_j)\varphi\, dy = \int (-\nabla^2)(s_n h_j) N\varphi\, dy.$$

So we have the inequality

$$\limsup_{j\to\infty} \lim_{n\to\infty} \int (-\nabla^2)(s_n h_j) N\varphi\, dy \le \int s\varphi\, dy.$$

A smooth superharmonic function has a negative Laplacian, so $\nabla^2 s_n \le 0$, and by our assumption, $N\varphi \ge 0$. Also $h_j \ge 0$. Therefore,

$$\begin{aligned}(-\nabla^2)(s_n h_j)N\varphi &\ge (-\nabla^2)(s_n h_j)N\varphi + \nabla^2(s_n)h_j N\varphi \\ &= -2\nabla\cdot(s_n\nabla(h_j)N\varphi) + 2s_n(\nabla h_j\cdot\nabla N\varphi) + s_n\nabla^2(h_j)N\varphi.\end{aligned}$$

The divergence goes away when we integrate, so

$$\int (-\nabla^2)(s_n h_j)N\varphi\, dy \ge \int 2s_n(\nabla h_j\cdot\nabla N\varphi) + s_n\nabla^2(h_j)N\varphi\, dy.$$

Our goal now is to choose h_j so that the integral is uniformly bounded above by C_0/j, where C_0 is some constant.

A bound on $\nabla N\varphi$ and $N\varphi$. For $y \in Q$, let $\delta = d(y, Q^c)$, and pick a point $x \in Q^c$ with $d(y,x) = \delta$. Our assumption says that $N\varphi$ is nonnegative everywhere and zero on Q^c, which means that the gradient is zero on Q^c. By Lemma 4.17 below, there is a constant $C > 0$ with

$$\left|\nabla N\varphi(y)\right| = \left|\nabla N\varphi(y) - \nabla N\varphi(x)\right| \le C\delta\log 1/\delta$$

if $\delta = |y-x|$ is sufficiently small. The constant depends only on the diameter of Q. The bound on the gradient implies a bound on $N\varphi$:

$$|N\varphi(y)| = |N\varphi(y) - N\varphi(x)| \le \int_0^\delta Ct\log 1/t\, dt \le C\delta^2\log 1/\delta,$$

where the last inequality is true if $\delta < 1/e$.

We choose test functions h_j as in Lemma 4.18. We have $|\nabla h_j| \le 1/(j\delta \log 1/\delta)$ and $|\nabla^2 h_j| \le |\partial^2 h_j/\partial y_1^2| + \cdots + |\partial^2 h_j/\partial y_d^2| \le 2d/(j\delta^2 \log 1/\delta)$. Therefore,

$$|\nabla h_j||\nabla N\varphi| \le C/j \quad \text{and} \quad |\nabla^2 h_j||N\varphi| \le 2Cd/j$$

and

$$\limsup_{n\to\infty} \left| \int 2s_n(\nabla h_j \cdot \nabla N\varphi)\, dy \right| \le \frac{2C}{j} \int_{h_j>0} |s|\, dy$$

$$\limsup_{n\to\infty} \left| \int s_n(\nabla^2 h_j N\varphi)\, dy \right| \le \frac{2Cd}{j} \int_{h_j>0} |s|\, dy.$$

Set $C_0 = 2(1+d)C$. Then, as promised earlier, $\int s_n h_j \varphi\, dy \ge -C_0/j$.

Take $n \to \infty$ and then take $j \to \infty$ to see that

$$\int s\varphi\, dy = -\lim_{j\to\infty} \lim_{n\to\infty} \int \nabla^2(s_n h_j) N\varphi\, dy \le 0$$

by the earlier calculation. □

We therefore finally have the theorem:

Theorem 4.16. *A properly supported weight function has a quadrature domain.*

Proof. Let Q be the set from Corollary 4.12. Then $N(w - \mathbb{1}_Q)$ is nonnegative and zero outside Q. By Theorem 4.15, for any integrable superharmonic function s on Q, $\int s(w - \mathbb{1}_Q)\, dy \ge 0$, so

$$\int sw\, dy \ge \int_Q s\, dy.$$

This is exactly the quadrature domain property, so Q is a quadrature domain. □

4.6. **Denouement 1: log-Lipschitz continuity.** We owe two lemmata that we must prove. First, a lemma about the modulus of continuity of the Newtonian potential, which we use to bound both the potential and its first derivative.

The lemma states, roughly, that $\nabla N\varphi$ is very close to being Lipschitz.

Lemma 4.17 (Günther, [2], §13)**.** *Suppose φ is bounded and measurable and zero outside a bounded set E. If $y, y' \in \mathbb{R}^d$ and $|y - y'| = \varepsilon$, then*

$$\left| \frac{\partial N\varphi}{\partial y_i}(y) - \frac{\partial N\varphi}{\partial y_i}(y') \right| = O\left(\varepsilon \log \frac{1}{\varepsilon} \right).$$

The constant in the O-notation depends only on $\max|\varphi|$ *and* $\operatorname{diam} E$.

Proof. Write both terms as derivatives of integrals, move them both under the same integral sign, and move the derivative inside the integral, to get

$$\left|\frac{\partial N\varphi}{\partial y_i}(y) - \frac{\partial N\varphi}{\partial y_i}(y')\right| = \left|\int_\Omega \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right] \varphi(x)\,dx\right|.$$

This is justified because $\partial G/\partial y_i$ is locally integrable and φ is bounded.

Let $A = \{x \in \Omega : |x-y| < 2\varepsilon\}$, and break the integral on the last line up into $\int_A$ and $\int_{E\setminus A}$. The first derivatives of $G(x,y)$ are $O(||x-y||^{1-d})$, so

$$\int_A \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right] \varphi(x)\,dx = O\left(\int_{B_{2\varepsilon}(y)} ||x-y||^{1-d}\,dx\right) = O(\varepsilon).$$

For the part outside of A, we estimate the integrand with derivatives. By the mean value theorem, there is a point y'' on the line segment between y and y' with $\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y') = (y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')$. The second derivatives of $G(x;y)$ are $O(||x-y||^{-d})$, so that dot product is at most

$$|y-y'| \times \left|\nabla\frac{\partial G}{\partial y_i}(x,y'')\right| = O(\varepsilon||x-y''||^{-d}).$$

When x is not in A, $||x-y|| \geq 2\varepsilon$, so

$$\begin{aligned} ||x-y''|| &\geq ||x-y|| - ||y-y''|| \\ &\geq ||x-y|| - ||y-y'|| \geq \tfrac{1}{2}||x-y||. \end{aligned}$$

Therefore $||x-y''||^{-d} \leq 2^d||x-y||$ and $O(||x-y''||^{-d}) = O(||x-y||^{-d})$, and we have the more concrete bound $|(y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')| = O(\varepsilon||x-y||^{-d})$, where the bound depends only on $\max|\varphi|$.

If $E\setminus A$ is empty, the integral over it is zero. Otherwise, let $\overline{B_{r_1}}\setminus B_{r_0}$ be the minimal closed annulus containing $E\setminus A$, i.e. $r_0 := \inf\{|x-y| : x \in E\setminus A\}$ and $r_1 := \sup\{|x-y| : x \in E\setminus A\}$, with $r_0 \geq 2\varepsilon$ and $r_1 \leq \operatorname{diam} E$.

We can estimate the integral over $E\setminus A$ by

$$\begin{aligned} \int_{E\setminus A} \left|(y-y')\cdot\nabla\frac{\partial G}{\partial y_i}(x,y'')\right| \varphi(x)\,dx &= \int_{E\setminus A} O(\varepsilon||x-y||^{-d})\,dx \\ &\leq O\left(\int_{r_0}^{r_1} \varepsilon r^{-d} C_d r^{d-1}\,dr\right) \\ &= O\left(\varepsilon \log\frac{r_1}{r_0}\right). \end{aligned}$$

The constants in the O-notation depend only on $\max|\varphi|$.

But we know that $r_1/r_0 \le \operatorname{diam} E/2\varepsilon = O(1/\varepsilon)$, so $O(\varepsilon \log r_1/r_0) = O(\log 1/\varepsilon)$. Combine the estimates for A and $E \setminus A$ to get the result:

$$\begin{aligned}\int_E \left[\frac{\partial G}{\partial y_i}(x,y) - \frac{\partial G}{\partial y_i}(x,y')\right] \varphi(x)\,dx &= \int_A (\cdots)\varphi(x)\,dx + \int_{E\setminus A} (\cdots)\varphi(x)\,dx \\ &= O(\varepsilon) + O\left(\varepsilon \log \frac{1}{\varepsilon}\right) \\ &= O\left(\varepsilon \log \frac{1}{\varepsilon}\right).\end{aligned}$$

This is the result that we want. □

4.7. **Denouement 2: a smooth approximation of the distance function.** The second lemma that we owe gives us smooth test functions with reasonable first and second derivatives.

Let $\xi(t) := 1/t(\log 1/t)$. Then $\xi'(t) = (1 - \log 1/t)\xi^2(t)$, so ξ is decreasing on $(0, 1/e)$, and $\int_0^{t_1} \xi(t)\,dt = \infty$ and $\int_0^{t_1} \xi'(t)\,dt = -\infty$ for any small t_1. Also, on the interval $(0, 1/e)$, we have the upper bound $|\xi'| \le \xi/t$.

Lemma 4.18 (Hedberg, [3], Lemma 4)**.** *Let δ denote $d(x, Q^c)$. If Q is a bounded open set, there is a sequence $h_j \in C_c^\infty(Q)$ with $0 \le h_j \le 1$ everywhere, $h_j(x) = 1$ if $\delta > 1/j$, and*

$$\left|\nabla h_j(x)\right| \le \frac{\xi(\delta)}{j}, \qquad \left|\frac{\partial^2}{\partial y_i \partial y_j} h_j(y)\right| \le \frac{2\xi(\delta)}{j\delta}.$$

Proof. Let $\Delta_Q(x)$ be the smooth distance function from Stein §VI.2 Theorem 2 [7], quoted below as Lemma 4.19. Let the constant in that lemma be called C.

Let $\mathrm{H}_j(t) := \int_0^t \eta_{1/Cj}(\tau)\,d\tau$, where η_ε is from Lemma 4.20. Our smooth test functions will be $h_j(x) := \mathrm{H}_j(\Delta_Q(x))$. These are compactly supported because $\Delta_Q \le C\delta$ and H_j is zero for small enough arguments.

The gradient of this function is $\eta_j(\Delta_Q)\nabla\Delta_Q$. The second derivatives are

$$\frac{\partial^2 h_j}{\partial x_i \partial x_j} = \eta_j(\Delta_Q)\frac{\partial^2 \Delta_Q}{\partial x_i \partial x_j} + \eta_j'(\Delta_Q)\frac{\partial \Delta_Q}{\partial x_i}\frac{\partial \Delta_Q}{\partial x_j}.$$

We plug in the bounds $|\eta_j(t)| \le \xi(t)/j$, $|\eta_j'(t)| \le |\xi'(t)|/j \le |\xi(t)|/jt$ from the first lemma below, and the bounds $\delta \le \Delta_Q$ and $|\nabla\Delta_Q| \le C$ and $|\partial_i \partial_j \Delta_Q| \le C/\delta$ from the second lemma below. The resulting bounds are C times greater than what we want, but we can get around that by multiplying j by $\lceil C \rceil$. □

Lemma 4.19 (Stein §VI.2 Theorem 2 [7]). *Let Q be an open set. There exists a smooth function $\Delta_Q(x)$ on Q with $d(x,Q^c) \le \Delta_Q(x) \le Cd(x,Q^c)$ for some positive constant $C > 0$, and*

$$|\nabla \Delta_Q| \le C \qquad \left|\frac{\partial^2}{\partial x_i \partial x_j}\Delta_Q\right| \le \frac{C}{d(x,Q^c)}.$$

Sketch of proof. Let the side length of a cube ω be denoted by side(ω). The set Q can be written as a union of closed cubes with disjoint interiors in such a way that the side length of each cube ω is a power of two, and

$$d(\omega,Q^c) \le 8\,\mathrm{side}(\omega) \le 4d(\omega,Q^c).$$

Scale up each cube around its center by a factor of $3/2$. All of the cubes are still contained in Q. If ω is a cube containing x, then $\mathrm{side}(\omega)/d(x,Q^c) \in [\frac{1}{13},4]$, and there are at most 2^d cubes of a certain size containing a point, so each point is contained in at most 6×2^d cubes.

Pick $h \ge 0$ smooth so that h is 1 on the unit cube and 0 on the cube of side length $3/2$. Scale and translate this to get a function h_ω for each cube ω in the decomposition which is 1 on ω and 0 on a slightly larger cube around ω. The scaling multiplies the first derivatives by a factor of $1/\,\mathrm{side}(\omega)$, and the second derivatives by a factor of $1/\,\mathrm{side}(\omega)^2$.

Let $\Delta_Q(x) := 13\sum_{\omega \text{ a cube}} \mathrm{side}(\omega) h_\omega(x)$. This sum is locally finite, and it's at least $d(x,Q^c)$. Let $C := 13\times 6\times 2^d \times (4 \vee \max|\nabla h| \vee 13\max|\partial_i\partial_j h|)$. Then $\Delta_Q(x) \le Cd(x,Q^c)$, and

$$|\nabla \Delta_Q| \le C, \qquad \left|\frac{\partial^2 \Delta_Q}{\partial x_i \partial x_j}\right| \le \frac{C}{d(x,Q^c)}$$

which is what we want to have. □

Lemma 4.20. *For $\varepsilon > 0$, there is a function $\eta_\varepsilon \in C_c^\infty(0,\varepsilon)$ with $0 \le \eta_\varepsilon \le \varepsilon\xi$ and $|\eta_\varepsilon'| \le \varepsilon|\xi'|$ and $\int \eta_\varepsilon\, dt = 1$.*

Proof. Let $\varepsilon < 1/e$ without loss of generality. Then ξ is decreasing on $(0,\varepsilon)$. Choose a sequence of smooth functions f_n on $(0,\varepsilon)$ that increase to $-\varepsilon\xi' \ge 0$ from below on that interval.

Let $F_n(t) := \int_t^\varepsilon f_n(\tau)\,d\tau$. Then $F_n \to \varepsilon(\xi(t) - \xi(\varepsilon))$ pointwise, and the integral of $\xi(t)$ on $[0,\varepsilon]$ is positive infinity, so $\int_0^\varepsilon F_n(t)\,dt \to \infty$. Let n be large enough that this integral is at least one. Then $\eta_j(t) := F_n(t)/\int_0^\varepsilon F_n(t)\,dt$ has the desired properties. □

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