WATSON'S THIRD INTEGRAL

Hannah Cairns, May 9, 2007

Contents

1. Introduction

This note is a collection of facts involved in the evaluation of the integral

$$I_3 = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{1}{3 - (\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}\, d\theta_1\, d\theta_2\, d\theta_3 \tag{1}$$

first obtained in (Watson 1939). It is the most difficult of the three integrals considered in that paper.

It will be used to find the probability that a random walk in three dimensions ever returns to the origin. Watson's motives for evaluating I_3 are somewhat different from ours. In his introduction, he says

> The desirability of investigating the triple integrals (...) has arisen as a consequence of their having appeared in a recent paper in ferromagnetic anisotropy by F. van Peype, a pupil of H. A. Kramers. The problem of evaluating them was proposed by Kramers to R. H. Fowler who communicated it to G. H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the journey to Birmingham without difficulty.

There is an expression in closed form for I_3, but to understand where it comes from we will need to enter a dark thicket: the theory of elliptic integrals. This theory should really be approached through theta functions. We do not intend to use this difficult and forbidding material. (We do not necessarily even understand it.) Instead, we use relatively elementary ideas from hypergeometric functions.

The argument Watson used has been left intact in section 3. It has been copied more or less verbatim from the paper mentioned above, although some errors have probably been introduced. Like most papers, though, Watson's paper is not self-contained and uses a variety of results with which the reader may not be familiar. So these results have been proved in sections 4 through 7.

Section 4 is an introduction to hypergeometric functions, and it proves some fundamental identities. Section 5 proves some properties of the elliptic integral of the first kind K; for example, a series for $K(\sqrt{1-x^2})$ around $x=0$ is worked out. Section 6 defines one of Appell's functions, F_4, and it derives an identity between F_4 and a product of two hypergeometric functions. Section 7 proves transformation formulae for the elliptic integral of the first kind, and it finds the value of two singular moduli, k_6 and $k_{2/3}$.

Note

The paper by W. F. Van Peype was published in *Physica* in 1938. See the bibliography. The paper is in German, which I can't read, and for that matter is about ferromagnetic crystals, which I don't understand.

However, as far as I could tell, although I_3 does arise, more importance seems to be attached to the integrals

$$\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos\theta_3}{3-\cos\theta_1-\cos\theta_2-\cos\theta_3}\,d\theta_1\,d\theta_2\,d\theta_3 \text{ and}$$
$$\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos\theta_2\cos\theta_3}{3-\cos\theta_1-\cos\theta_2-\cos\theta_3}\,d\theta_1\,d\theta_2\,d\theta_3,$$

that is, other Fourier coefficients of the integrand. It is easy to see that the first integral is $I_3 - 1/3$.

I'm not sure when closed forms for these coefficients were obtained, although according to (Joyce and Zucker) there are some.

2. What good is I_3?

The integral

$$3I_3 = \int_0^\pi \int_0^\pi \int_0^\pi \frac{1}{1 - \frac{1}{3}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}\, d\theta_1\, d\theta_2\, d\theta_3$$

gives the average number of returns to the origin for the symmetric random walk in three dimensions.

Let F be the generating function for one step of a three-dimensional random walk on the simple cubic lattice, $\mathbb{Z}^3$.

$$F(x, y, z) = \frac{1}{6}(x + x^{-1} + y + y^{-1} + z + z^{-1}).$$

Each possible step has probability $\frac{1}{6}$. Since the steps are independent, the distribution after two steps is F^2. In general, after the walk has taken m steps, the distribution is F^m. The coefficient $x^i y^j z^k$ of $F(x, y, z)^m$ is the probability that the walk has reached (i, j, k) at the time m.

Let u_m be the probability that the random walk returns to the origin at m. Substitute $x = e^{i\theta_1}$, $y = e^{i\theta_2}$, $z = e^{i\theta_3}$. By orthogonality,

$$\begin{aligned} u_m &= \text{constant term of } F(x, y, z)^m \\ &= \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F\left(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\right)^m d\theta_1\, d\theta_2\, d\theta_3. \end{aligned}$$

The reader will notice that, in these variables,

$$F = \frac{1}{3}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3).$$

The expected number of returns to the origin (including the one at time 0), is the sum of all probabilities u_m. For the proof, see chapter 13 of (Feller).Alternatively, one can reason as follows. If I_m is the function which is 1 if there's a return at time m, and 0 otherwise, then by monotone convergence,

$$\mathrm{E}[\#\text{ returns}] = \mathrm{E}\left[\sum_{m=0}^{\infty} I_m\right] = \sum_{m=0}^{\infty} E[I_m] = \sum_{m=0}^{\infty} u_m.$$

Since finite sums commute with the integral sign,

$$\sum_{m=0}^{\infty} u_m = \lim_{M\to\infty} \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1 - F^{M+1}}{1 - F}\, d\theta_1\, d\theta_2\, d\theta_3.$$

The absolute value of F is less than 1, except in two places, $\theta = (0, 0, 0)$ and $\theta = (\pi, \pi, \pi)$. Therefore, the integrands are all dominated by $|2/(1 - F)|$ and converge pointwise almost everywhere. Since $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 3 - \frac{1}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mathrm{O}(||\theta||^3)$,

$$\frac{2}{1 - F} = \frac{12}{\theta_1^2 + \theta_2^2 + \theta_3^2} + \mathrm{O}(||\theta||^{-1}).$$

But in three dimensions, the integral of r^{-2} in a bounded region is finite. The integral of the dominating function near zero is less than infinity.

$$\int_{||\theta|| \le \varepsilon} \frac{2}{1 - F}\, d\theta_1\, d\theta_2\, d\theta_3 \approx \int_{||\theta|| \le \varepsilon} \frac{12}{\theta_1^2 + \theta_2^2 + \theta_3^2}\, d\theta_1\, d\theta_2\, d\theta_3 = \int_{r \le \varepsilon} 48\pi\, dr < \infty.$$

But $2/(1-F)$ is bounded outside the region $||\theta|| < \varepsilon$. So

$$\int \left| \frac{2}{1-F} \right| d\theta_1\, d\theta_2\, d\theta_3 < \infty$$

since F is not equal to 1 anywhere else.

The dominated convergence theorem applies, and we conclude that

$$\sum_{m=0}^{\infty} u_m = \frac{1}{(2\pi)^3} \int \frac{1}{1-F}\, d\theta_1\, d\theta_2\, d\theta_3 = 3I_3.$$

We can obtain the probability of return to the origin as follows. Let this probability be f. The "return" at time zero has probability 1. A second return occurs with probability f, a third occurs with probability f^2, and so on. The average number of returns is

$$\begin{aligned} E[\#\text{ returns}] &= \sum_{m=1}^{\infty} m\, \mathrm{P}[\text{the walk makes exactly } m \text{ returns}] \\ &= \sum_{m=1}^{\infty} \mathrm{P}[\text{the walk makes } m \text{ returns or more}]. \end{aligned}$$

This is

$$U = 1 + f + f^2 + \cdots = \frac{1}{1-f}.$$

Consequently, $f = (U-1)/U = (3I_3 - 1)/3I_3$.

We now attempt to evaluate I_3 in closed form.

3. Watson's argument

$$I_3 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{1}{3-(\cos\theta_1+\cos\theta_2+\cos\theta_3)}\,d\theta_1\,d\theta_2\,d\theta_3.$$

We begin by setting $x = \tan\theta_1/2$, $y = \tan\theta_2/2$, $z = \tan\theta_3/2$. This coordinate substitution makes the integrand rational. If $x = \tan\theta/2$, then

$$\frac{d\theta}{dx} = \frac{2}{1+x^2} \qquad \cos\theta = \frac{1-x^2}{1+x^2} \qquad \sin\theta = \frac{2x}{1+x^2}$$

For example,

$$3-(\cos\theta_1+\cos\theta_2+\cos\theta_3) = \frac{2x^2+2y^2+2z^2+4x^2y^2+4x^2z^2+4y^2z^2+6x^2y^2z^2}{(1+x^2)(1+y^2)(1+z^2)}.$$

We then convert to polar coordinates r, θ, φ,

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta \qquad J = r^2\sin\theta.$$

There will be a symmetry around $\psi = \pi/2$, and

$$\int_0^{\pi/2}(\cdots)d\varphi = \frac{1}{2}\int_0^{\pi}(\cdots)d\psi = \int_0^{\pi/2}(\cdots)d\psi.$$

These operations give us

$$(1) = \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(x^2+y^2+z^2)+2(x^2y^2+x^2z^2+y^2z^2)+3x^2y^2z^2}\,dx\,dy\,dz$$

$$= \frac{4}{\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^\infty \frac{\sin\theta}{1+2r^2\sin^2\theta(\cos^2\theta+\sin^2\theta\sin^2\varphi\cos^2\varphi)+3r^4\sin^4\theta\cos^2\theta\sin^2\varphi\cos^2\varphi}\,dr\,d\theta\,d\varphi$$

$$= \frac{4}{\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^\infty \frac{\sin\theta}{1+2r^2\sin^2\theta(\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2\psi)+\frac{3}{4}r^4\sin^4\theta\cos^2\theta\sin^2\psi}\,dr\,d\theta\,d\psi.$$

Now replace r by a new variable t defined by the equation $r\sin\theta = t\sqrt{2}$, and integrate over θ.

$$(1) = \frac{4\sqrt{2}}{\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^\infty \frac{1}{1+t^2(4\cos^2\theta+\sin^2\theta\sin^2\psi)+3t^4\cos^2\theta\sin^2\psi}\,dt\,d\theta\,d\psi$$

We know that, under the substitution $\tan\theta = x$,

$$\int_0^{\pi/2}\frac{1}{a+b\cos^2\theta}d\theta = \int_0^\infty \frac{1}{a(1+x^2)+b}dx = \frac{\pi/2}{\sqrt{a+b}\sqrt{a}}.$$

So

$$(1) = \frac{2\sqrt{2}}{\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{1}{\sqrt{1+4t^2+3t^4\sin^2\psi}\sqrt{1+t^2\sin^2\psi}}\,dtd\psi$$

Next substitute $\tan\psi = \xi$, and then write $\xi = \zeta/(1+t^2)$ and integrate with respect to t.

$$(1) = \frac{2\sqrt{2}}{\pi^2}\int_0^\infty\int_0^\infty \frac{1}{\sqrt{1+\xi^2+4t^2+4t^2\xi^2+3t^4\xi^2}\sqrt{1+\xi^2+t^2\xi^2}}\,dt\,d\xi$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^\infty\int_0^\infty \frac{1}{\sqrt{1+t^2}\sqrt{1+4t^2+\zeta^2+3t^2\zeta^2}\sqrt{1+\zeta^2}}\,dt\,d\zeta$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^\infty K'\left(\frac{1+\zeta^2}{4+3\zeta^2}\right)\frac{1}{\sqrt{4+3\zeta^2}\sqrt{1+\zeta^2}}\,d\zeta$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^{\pi/2} K'\left(\frac{1}{\sqrt{4-\sin^2\chi}}\right)\frac{1}{\sqrt{4-\sin^2\chi}}\,d\chi$$

where $\zeta = \tan\chi$. See section 5.2 for the definition of K'.

In section 5.4 we prove that

$$K'(k) = -\frac{1}{2\pi}\left[\frac{d}{dz}\sum_{m=0}^{\infty}\frac{\Gamma^2(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)m!}k^{2m+2z}\right]_{z=0}.$$

It follows that

$$(1) = -\frac{\sqrt{2}}{\pi^3}\int_0^{\pi/2}\left[\frac{d}{dz}\sum_{m=0}^{\infty}\frac{\Gamma^2(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)m!}\frac{1}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}}\right]_{z=0}d\chi$$

$$= -\frac{\sqrt{2}}{\pi^3}\left[\frac{d}{dz}\sum_{m=0}^{\infty}\frac{\Gamma^2(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)m!}\int_0^{\pi/2}\frac{1}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}}\,d\chi\right]_{z=0}$$

We have to move the derivative and the sum through the integral. The justification is left to the reader. Note that $4-\sin^2\chi$ only varies between 4 and 3, so the integrand can be written as $f(0,\chi)+zf_z(0,\chi)+o(z)$, where $o(z)$ is uniform in the other variable χ.

Consider $(4-\sin^2\chi) = \frac{1}{2}(7+\cos 2\chi)$. Choose c to be a root of $c^2+1 = 14c$. The possible values for c are $7-4\sqrt{3}$ and $7+4\sqrt{3}$.

Let $c = 7-4\sqrt{3} = (2-\sqrt{3})^2$. To four decimal places, $c \approx 0.0718$. In fact c is so small that

$$\sqrt{4c}+\sqrt{c^2} < 1.$$

Now replace $(4-\sin^2\chi)$ with

$$\frac{1}{2}(7+\cos 2\chi) = \frac{1}{4c}(1+c^2+2c\cos 2\chi) = \frac{1}{4c}(1+ce^{2\chi i})(1+ce^{-2\chi i}).$$

The integrand becomes

$$\frac{1}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}} = \frac{(4c)^{m+z+\frac{1}{2}}}{((1+ce^{2\chi i})(1+ce^{-2\chi i}))^{m+z+\frac{1}{2}}}$$

Expand each factor in the denominator in a series of ascending powers of c, multiply the two series together, and integrate.

Only the terms with no factor $e^{2m\chi i}$ will contribute; the others will vanish when the integral is performed (after terms $e^{+2m\chi i}$ are paired with $e^{-2m\chi i}$).

So,

$$\int_0^{\pi/2} \frac{d\chi}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}} = \frac{\pi}{2}(4c)^{m+z+\frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{(m+z+\frac{1}{2})_j}{j!} \right)^2 c^{2j} \tag{2}$$

We are using the Pochhammer symbol $(x)_j = (x)(x+1)\cdots(x+j-1)$.
Substitute this into the above expression.

$$\begin{aligned}
&\frac{\pi}{2}\sum_{m=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)m!}(4c)^{m+z+\frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{(m+z+\frac{1}{2})_j}{j!} \right)^2 c^{2j} \\
&\qquad = \frac{\pi}{2}\frac{\Gamma^2(\frac{1}{2}+z)}{\Gamma(1+2z)}(4c)^{z+\frac{1}{2}} \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(z+\frac{1}{2})_{m+j}(z+\frac{1}{2})_{m+j}}{(1+2z)_m(1)_j}(4c)^m(c^2)^j \\
&\qquad\qquad = \frac{\pi}{2}\frac{\Gamma^2(\frac{1}{2}+z)}{\Gamma(1+2z)}(4c)^{z+\frac{1}{2}} F_4(\frac{1}{2}+z, \frac{1}{2}+z; 1+2z, 1; 4c, c^2).
\end{aligned}$$

This function F_4 is one of several defined by Appell. They are generalizations of the hypergeometric function to the two-variable case. See (Slater), Chapter 8.

$$F_4(a,b;c,c';x,y) = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{m+j}(b)_{m+j}}{(c)_m(c')_j} x^m y^j$$

This function converges at least when $|x|^{1/2}+|y|^{1/2} < 1$.

We have the following relation. If $a+b+1$ is equal to $c+c'$, and u and v are sufficiently small, then

$$F_4(a,b;c,c';u(1-v),v(1-u)) = {}_2F_1(a,b;c;u)\,{}_2F_1(a,b;c',v).$$

This is proved in section 6, using certain well-known identities about hypergeometric functions, which are themselves proved in section 4. The above formula is true at least when $|u|$ and $|v|$ are less than $\frac{1}{2}$, and the arguments of F_4 are in the domain

$$|u(1-v)|^{1/2} + |v(1-u)|^{1/2} < 1.$$

The function

$$F_4\Big(\tfrac{1}{2}+z, \tfrac{1}{2}+z; 1+2z, 1; 4c, c^2\Big).$$

does meet the requirement that $a+b+1 = c+c'$.

We indicated earlier that $\sqrt{4c}+\sqrt{c^2} < 1$, so we just need values of u and v such that

$$u(1-v) = 4c \qquad v(1-u) = c^2.$$

These conditions can be satisfied by the following u and v.

$$1-u = (2-\sqrt{3})^2(\sqrt{3}+\sqrt{2})^2 \approx 0.7107.$$

$$v = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2 \approx 0.00725.$$

We readily check that $v(1-u) = c^2$ and $u-v = 4c-c^2 = 1-10c$.

Hence

$$\begin{aligned}
&F_4\Big(\tfrac{1}{2}+z, \tfrac{1}{2}+z; 1+2z, 1; 4c, c^2\Big) \\
&\qquad = {}_2F_1\left(\begin{matrix} \frac{1}{2}+z, \frac{1}{2}+z \\ 1+2z \end{matrix} \middle| u \right) {}_2F_1\left(\begin{matrix} \frac{1}{2}+z, \frac{1}{2}+z \\ 1 \end{matrix} \middle| v \right).
\end{aligned}$$

On page 3 and 4, we expressed (1) in the following peculiar terms.

$$-\frac{\sqrt{2}}{\pi^3}\left[\frac{d}{dz}\sum_{m=0}^{\infty}\frac{\Gamma^2(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)m!}\int_0^{\pi/2}\frac{1}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}}\,d\chi\right]_{z=0}.$$

We substituted the following expansion, which we obtained from the factorization of $(4-\sin^2\chi)$ into $(4c)^{-1}(1+ce^{2\chi i})(1+ce^{-2\chi i})$, into the expression above.

$$(2)\qquad \int_0^{\pi/2}\frac{d\chi}{(4-\sin^2\chi)^{m+z+\frac{1}{2}}}=\frac{\pi}{2}(4c)^{m+z+\frac{1}{2}}\sum_{j=0}^{\infty}\left(\frac{(m+z+\frac{1}{2})_j}{j!}\right)^2 c^{2j}.$$

This gave

$$(1)=-\frac{1}{(\sqrt{2})\pi^2}\left[\frac{d}{dz}\frac{\Gamma^2(\frac{1}{2}+z)}{\Gamma(1+2z)}(4c)^{z+\frac{1}{2}}F_4\left(\frac{1}{2}+z,\frac{1}{2}+z;1+2z,1;4c,c^2\right)\right]_{z=0}$$

$$=-\frac{1}{(\sqrt{2})\pi^2}\left[\frac{d}{dz}\frac{\Gamma^2(\frac{1}{2}+z)}{\Gamma(1+2z)}(4c)^{z+\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{1}{2}+z,\frac{1}{2}+z\\1+2z\end{matrix}\middle|\,u\right){}_2F_1\left(\begin{matrix}\frac{1}{2}+z,\frac{1}{2}+z\\1\end{matrix}\middle|\,v\right)\right]_{z=0}$$

Kummer's transformation, which was proved in section 4, gives us the following identity.

$${}_2F_1\left(\begin{matrix}\frac{1}{2}+z,\frac{1}{2}+z\\1\end{matrix}\middle|\,v\right)=(1-v)^{-z-\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{1}{2}+z,\frac{1}{2}-z\\1\end{matrix}\middle|\,\frac{v}{v-1}\right)$$

for $v<1/2$. The hypergeometric function on the right-hand side is even, so its derivative at $z=0$ vanishes.

We can reverse this identity after the ${}_2F_1$ is outside the derivative.

$${}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|\,\frac{v}{v-1}\right)=(1-v)^{\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|\,v\right)=(1-v)^{\frac{1}{2}}K(\sqrt{v}).$$

Therefore, since $4c=u(1-v)$,

$$(1)=-\frac{(1-v)^{\frac{1}{2}}\sqrt{2}}{\pi^3}K(\sqrt{v})\left[\frac{d}{dz}\frac{\Gamma^2(\frac{1}{2}+z)}{\Gamma(1+2z)}u^{z+\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{1}{2}+z,\frac{1}{2}+z\\1+2z\end{matrix}\middle|\,u\right)\right]_{z=0}$$

$$=\frac{(4c)^{\frac{1}{2}}2\sqrt{2}}{\pi^2}K(\sqrt{v})K'(\sqrt{u}).$$

We might be satisfied with this. We have represented the integral I_3 in terms of two elliptic functions.

$$I_3=\frac{4\sqrt{2}(2-\sqrt{3})}{\pi^2}K\big((2-\sqrt{3})(\sqrt{3}-\sqrt{2})\big)K\big((2-\sqrt{3})(\sqrt{3}+\sqrt{2})\big).$$

$$3I_3=1.516386059151978018156012159+.$$

But we can write I_3 in terms of $K(\sqrt{v})^2$, although this makes the algebraic part a bit more complicated. And since we are able to, we might as well do it.

Now the *really* hard part. Let $\tau(k)$ be defined as in section 7,

$$\tau(k)=\frac{K'}{K}(k).$$

We prove in section 7 that, when $\sqrt{1-u}=(2-\sqrt{3})(\sqrt{3}+\sqrt{2})$,

$$\tau(\sqrt{u})=\frac{1}{\tau(\sqrt{1-u})}=\sqrt{\frac{3}{2}}.$$

We also showed that $\tau(\sqrt{v}) = \sqrt{6}$. Therefore,

$$\tau(\sqrt{v}) = 2\tau(\sqrt{u}).$$

But if $\tau(a) = \tau(b)$, then $a = b$ (when $0 < a, b < 1$). The lower-right formula from section 7, together with the equation on the bottom of the first page, gives

$$K(\sqrt{u}) = (1 + \sqrt{v})K(\sqrt{v}).$$

Since $\tau(\sqrt{1-u}) = \sqrt{2/3}$,

$$K(\sqrt{1-u}) = \sqrt{\frac{3}{2}}K(\sqrt{u}).$$

So,

$$K(\sqrt{1-u}) = \sqrt{\frac{3}{2}}K(\sqrt{u}) = \sqrt{\frac{3}{2}}(1 + \sqrt{v})K(\sqrt{v}).$$

Therefore,

$$\begin{aligned} I_3 &= \frac{(4c)^{\frac{1}{2}}2(1+\sqrt{v})\sqrt{3}}{\pi^2} K\big((2-\sqrt{3})(\sqrt{3}-\sqrt{2})\big)^2 \\ &= (18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6})\left(\frac{2K(\sqrt{v})}{\pi}\right)^2. \end{aligned}$$

This is our final expression for I_3. It is possible to write $K(\sqrt{v})$ as a product of gamma functions, but this was not proved until 1967. Watson himself ended the paper here, remarking only that the formula

$$\frac{2K(\sqrt{v})}{\pi} = \vartheta_3(e^{-\pi\sqrt{6}})^2 = (1 + 2e^{-\pi\sqrt{6}} + 2e^{-4\pi\sqrt{6}} + 2e^{-9\pi\sqrt{6}} + \cdots)^2$$

was a convenient way to start computing I_3.

The formula in terms of gamma functions can be written very nicely thanks to (Borwein and Zucker),

$$3I_3 = \frac{\sqrt{3}-1}{32\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2.$$

This is difficult to prove, but the paper by Borwein and Zucker is a good place to start. It all depends on an expression from analytic number theory; see (Selberg and Chowla).

(I am planning to write a sequel when I find out how this works.)

Also see (Joyce and Zucker), who via a paper by (Iwata)[1] were able to find a closed form for

$$W_2(z) = \frac{1}{\pi^3}\int_0^\pi \frac{1}{1 - \frac{z}{3}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}\,d\theta_1\,d\theta_2\,d\theta_3,$$

which is the generating function of $U(z) = \sum u_m z^m$.

[1]The *Natural Science Report of Ochanomizu University* appears to be hard to find, but the argument is recapitulated in Joyce and Zucker's paper.

4. Hypergeometric functions

Everyone should know a few theorems about hypergeometric functions. We are going to define the hypergeometric function on three parameters, and then prove some things about it.

A *hypergeometric series* is an infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ such that the ratio between successive terms, σ_{m+1}/σ_m, is a rational function of m.

Let x be a constant and $P(m)$ and $Q(m)$ be two monic polynomials, chosen so that

$$\frac{\sigma_{m+1}}{\sigma_m} = R(m) = x\frac{P(m)}{Q(m)} = x\frac{(m+a_1)\cdots(m+a_p)}{(m+b_1)\cdots(m+b_q)}.$$

The letters p and q denote the degrees of the numerator and denominator. We are generally going to put $\sigma_0 = 1$.

A *hypergeometric function* is a power series

$$\sum_{m=0}^{\infty} a_m x^m = 1 + \frac{P(0)}{Q(0)}\frac{x}{1!} + \frac{P(0)}{Q(0)}\frac{P(1)}{Q(1)}\frac{x^2}{2!} + \cdots$$

with hypergeometric coefficients. It is denoted by

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\,\middle|\, x\right).$$

The factorial in the denominator acts like an invisible $b_{q+1} = 1$. We can cancel it out by adding an $a_{p+1} = 1$, but why is it there in the first place?

The derivative of a ${}_pF_q$ is

$$\begin{aligned}\frac{d}{dx}{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) &= \frac{P(0)}{Q(0)} + \frac{P(0)}{Q(0)}\frac{P(1)}{Q(1)}\frac{x}{1!} + \frac{P(0)}{Q(0)}\frac{P(1)}{Q(1)}\frac{P(2)}{Q(2)}\frac{x^2}{2!} + \cdots \\ &= \frac{a_1\cdots a_p}{b_1\cdots b_q}\,{}_pF_q\left(\begin{matrix} a_1+1, \ldots, a_p+1 \\ b_1+1, \ldots, b_q+1 \end{matrix}\right)\end{aligned}$$

The factorial eats the inconvenient factor of $m+1$, which would otherwise increase the number of parameters. This pleasant expression for the derivative leads naturally to a differential equation.

Of course this is the standard notation and we would have to use it whether it made sense or not.

The series ${}_pF_q$ converges absolutely everywhere when $p \le q$. If p is equal to $q+1$, then it converges absolutely when $|x|$ is strictly less than 1, by the ratio test.

This is a very general function, and many other transcendental functions can be expressed by combining ${}_pF_q$ with simpler functions.[2]

In this note the function ${}_2F_1$ shows up a lot, and we had better prove some things about it. We have

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, x\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m\, m!} x^m$$

where $(a)_m = (a)(a+1)\cdots(a+m-1)$ is called the Pochhammer symbol.

---

[2]Try typing your favourite special function into Maple, and asking it to convert to ${}_pF_q$-s. The command for this is

```
convert(⟨function⟩, hypergeom);
```

Theorem. This series solves the *hypergeometric differential equation*

$$x(1-x)\frac{d^2f}{dx^2} + (c-(a+b+1)x)\frac{df}{dx} - abf = 0.$$

Proof. Let ∂_x denote the derivative on x. Let δ denote the operator $x\partial_x$, which acts on power series by sending x^α to αx^α.

We claim that the function $F = {}_2F_1(a,b;c;x)$ solves

$$\Big(x(1-x)\partial_x\partial_x + (c-(a+b+1)x)\partial_x - ab\Big)\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\,x\right) = 0$$

But this is the same as

$$\big[(\delta+c)\partial_x - (\delta+a)(\delta+b)\big]F = 0$$

or

$$(\delta+c)\partial_x F = (\delta+a)(\delta+b)F.$$

A series $\sum_m \sigma_m x^m$ will solve the above equation if

$$(m+c)(m+1)\sigma_{m+1} = (m+a)(m+b)\sigma_m$$

for all integer m, where we will set $\sigma_m = 0$ if $m < 0$.

By definition, the coefficients of ${}_2F_1(a,b;c;x)$ solve this equation for $m \geq 0$. When $m \leq -2$, both σ_m and σ_{m+1} are zero. When $m = -1$, the left side of the equation is zero since $m+1 = 0$, and the right side is zero since $\sigma_{-1} = 0$.

In a sense, the presence of the factor $m+1$ above allows the series to start at x^0. Could the series also start at x^{1-c}?

Yes; the shifted hypergeometric function

$$\sum_{m=0}^{\infty} \frac{(a+1-c)_m(b+1-c)_m}{(2-c)_m m!} x^{m+1-c} = x^{1-c}{}_2F_1\left(\begin{matrix}a+1-c,\ b+1-c\\2-c\end{matrix}\middle|\,x\right)$$

is also a solution of the above differential equation, valid on the interval $(0,1)$.

A second-order linear differential equation has exactly two linearly independent solutions, and unless $c = 1$ we now know every solution for the hypergeometric differential equation. Unfortunately, $c = 1$ is exactly the case we are interested in; more about this later.

4.1. **Vandermonde's identity.** We can evaluate a ${}_2F_1$ at $x = 1$. One way to do it is by using the integral representation for the ${}_2F_1$.

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\,x\right) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(a)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-b}\,dt$$

This is valid when $|x|$ is less than 1 and $\mathrm{Re}(c-b) > 0$, $\mathrm{Re}(b) > 0$.

However, since we don't need the full generality, we can prove the following form of the theorem in an elementary way. The general theorem is

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\,1\right) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}.$$

Theorem. Let $n \geq 0$ be an integer.

$${}_2F_1\left(\begin{matrix}a,-n\\c\end{matrix}\middle|\,1\right) = \frac{(c-a)_n}{(c)_n}.$$

PROOF. We start with the identity

$$\sum_{j=0}^{n}\binom{r}{n-j}\binom{s}{j} = \binom{r+s}{n} = \frac{(r+s-n+1)_n}{n!}.$$

This can be proven easily from $(1+x)^r(1+x)^s = (1+x)^{r+s}$.

The sum on the left is hypergeometric. The ratio between successive terms is

$$\binom{r}{n-j-1}\binom{s}{j+1}\Big/\binom{r}{n-j}\binom{s}{j} = \frac{(s-j)(n-j)}{(r-n+j+1)(j+1)}.$$

It follows that the sum is

$$\begin{aligned}\sum_{j=0}^{n}\binom{r}{n-j}\binom{s}{j} &= \binom{r}{n}\sum_{j=0}^{n}\frac{(-n)_j(-s)_j}{j!(r-n+1)_j}\\ &= \frac{(r-n+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n,-s\\ r-n+1\end{matrix}\right| 1\right)\end{aligned}$$

Now set $c = r-n+1$ and $a = -s$ and we are done with the proof.

$${}_2F_1\left(\left.\begin{matrix}-n,-s\\ r-n+1\end{matrix}\right| 1\right) = \frac{(r+s-n+1)_n}{(r-n+1)_n} \qquad \text{or} \qquad {}_2F_1\left(\left.\begin{matrix}-n,a\\ c\end{matrix}\right| 1\right) = \frac{(c-a)_n}{(c)_n}.$$

□

4.2. **Kummer's transformation.**

THEOREM.

$${}_2F_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\right| x\right) = (1-x)^{-a}\,{}_2F_1\left(\left.\begin{matrix}a,c-b\\ c\end{matrix}\right| \frac{x}{x-1}\right)$$

PROOF. Take the coefficient of x^m from the right-hand side. It is

$$\begin{aligned}[x^m]\sum_{j=0}^{\infty}\frac{(a)_j\,(c-b)_j}{(c)_j\,j!}\,\frac{x^j}{(1-x)^{j+a}}(-1)^j &= \sum_{j=0}^{m}(-1)^j\frac{(a)_j\,(c-b)_j\,(a+j)_{m-j}}{(c)_j\,j!\,(m-j)!}\\ &= \frac{(a)_m}{m!}\,{}_2F_1\left(\left.\begin{matrix}c-b,-m\\ c\end{matrix}\right| 1\right)\\ &= \frac{(a)_m\,(b)_m}{(c)_m\,m!}\end{aligned}$$

Hence the two sides are equal where they converge. □

4.3. **The gamma function.** Let $\Gamma(x)$ be the gamma function,

$$\Gamma(x) = \lim_{m\to\infty}\frac{m!\,m^{x-1}}{(x)_m}.$$

Here are a few of its familiar properties.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad \text{if } x > 0. \qquad\qquad x\Gamma(x) = \Gamma(x+1).$$

$$\frac{\Gamma(x+m)}{\Gamma(x)} = (x)(x+1)\cdots(x+m-1) = (x)_m.$$

$$\int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Setting $t = \sin^2\theta$,

$$2\int_0^{\pi/2} \sin(\theta)^{2a-1}\cos(\theta)^{2b-1}\,d\theta = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The identity $2\cos\theta\sin\theta = \sin 2\theta$ gives the multiplication formula

$$\Gamma(2x)\Gamma(1/2) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2).$$

The digamma function may be less familiar.

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx}\log\Gamma(x) = \lim_{m\to\infty}\left(\log m - \frac{1}{x} - \frac{1}{x+1} - \cdots - \frac{1}{x+m-1}\right).$$

The limit is a uniform limit of analytic functions, so the derivative and limit can be interchanged. The reader who has heard of Euler's constant γ will notice that $\psi(1)$ is equal to $-\gamma$.

The letter digamma is an obsolete Greek letter, which looked something like the Roman letter '*f*'. We use the letter ψ instead.

We will later want to know the value of $\psi(1) - \psi(\frac{1}{2})$. This can be worked out as follows.

$$\begin{aligned}
\psi(\tfrac{1}{2}) &= \lim_{m\to\infty}\left(\log m - 2 - \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \cdots - \frac{2}{2m-1}\right)\\
&= \lim_{m\to\infty}\left(\log(2m) - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{2m}\right) - \log 2\\
&\qquad - \lim_{m\to\infty}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2m}\right)\\
&= \psi(1) - \log 4.
\end{aligned}$$

In fact, every value $\psi(\frac{p}{q})$ with p and q rational can be expressed in terms of γ and logarithms.

One other useful fact is that, since $\Gamma(x+m) = (x)_m\Gamma(x)$,

$$\psi(x+m) = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+m-1} + \psi(x).$$

Hence,

$$\frac{\partial}{\partial c}(c)_m = (\psi(c+m) - \psi(c))(c)_m.$$

This shows us how to differentiate a ${}_pF_q$ on a parameter. For example,

$$\frac{\partial}{\partial c}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\,x\right) = \sum_{m=0}^{\infty}(\psi(c) - \psi(c+m))\frac{(a)_m\,(b)_m}{(c)_m}\frac{x^m}{m!}.$$

Recall that the differential equation

$$(\delta + c)\partial f = (\delta + a)(\delta + b)f$$

is solved by the following two functions in $0 < x < 1$.

$$S_1 = {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\,x\right) \qquad S_2 = x^{1-c}{}_2F_1\left(\begin{matrix}a+1-c,b+1-c\\2-c\end{matrix}\middle|\,x\right).$$

Consider $f_c(x) = (S_2 - S_1)/(c-1)$. It is clear that

$$(\delta + c)\partial f_c - (\delta + a)(\delta + b)f_c = 0.$$

We may take the limit as c approaches 1. We start by writing out the series,

$$f_c(x) = \sum_{m=0}^{\infty} \left(\frac{(a)_m\,(b)_m}{(c)_m} - x^{1-c} \frac{(a+1-c)_m\,(b+1-c)_m}{(2-c)_m} \right) x^m.$$

Since $\psi(c+m)$ increases fairly slowly, the expression above is uniformly convergent for x in a closed set inside $(0,1)$ and c fairly close to 1. Therefore, the derivative goes through the sum.

$$f_1(x) = \frac{\partial}{\partial c} f_c(x)\Big]_{c=1} = \sum_{m=0}^{\infty} \frac{\partial}{\partial c}\Big(\cdots\Big)\Big]_{c=1} x^m,$$

where $(\cdots)$ is the expression in parentheses above. Therefore, the coefficients of f_1 solve the same recurrence as the coefficients of f_c when $c \neq 1$.

$$(\delta+1)\partial f_1 - (\delta+a)(\delta+b) f_1 = 0.$$

If f_1 isn't proportional to S_1, then it's a new solution. We can use any pair of solutions S_1, S_2, not just the two above, provided that they are equal at $c = 1$.

This line of thought is continued in Section 5.3.

5. Formulae for elliptic integrals

The elliptic integral of the first kind, K, is

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-k^2x^2}} \frac{1}{\sqrt{1-x^2}}\, dx.$$

Set $x = \sin\theta$. The new value of $K(k)$ is

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2(\sin\theta)^2}}\, dx.$$

5.1. $K(\sqrt{k})$ is hypergeometric.

Expand the first term by the binomial theorem for

$$\begin{aligned} K(k) &= \sum_{j=0}^{\infty} \frac{(\frac{1}{2})_j}{j!} k^{2j} \int_0^1 \frac{x^{2j}}{\sqrt{1-x^2}}\, dx \\ &= \frac{1}{2}\sum_{j=0}^{\infty} \frac{(\frac{1}{2})_j}{j!} k^{2j} \frac{\Gamma(j+1/2)\Gamma(1/2)}{\Gamma(j+1)} \\ &= \frac{\pi}{2}\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}\right| k^2\right) \end{aligned}$$

The expression on the second line was obtained by making the change of variables $x \to \sqrt{x}$; this produced a beta integral.

In the third line, two identities were used: $\Gamma(1/2)^2$ is π, and

$$\Gamma\left(j+\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(\frac{1}{2}+j-1\right)\Gamma\left(\frac{1}{2}\right).$$

□

5.2. The conjugate, K'.

In section 3 we want to evaluate

$$\int_0^{\infty} \frac{1}{\sqrt{1+x^2}\sqrt{a^2+x^2}}\, dx.$$

Set $x = \tan\theta$. The integral above becomes

$$\int_0^{\infty} \frac{1}{\sec^2\theta\sqrt{a^2\cos^2\theta+\sin^2\theta}}\, dx = \int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2\theta+\sin^2\theta}}\, d\theta.$$

If $0 \le a^2 \le 1$, then the latter integral is clearly

$$\int_0^{\pi/2} \frac{1}{\sqrt{1+(a^2-1)\cos^2\theta}}\, d\theta = \int_0^{\pi/2} \frac{1}{\sqrt{1+(a^2-1)\sin^2\theta}}\, d\theta.$$

which is $K(\sqrt{1-a^2})$.

We give this function the name $K'(k) = K(\sqrt{1-k^2})$. We will find a power series for it.

5.3. **K' near zero.**

The function $K' = K(\sqrt{1-k^2})$ can be evaluated near zero.

$$K' \approx \ln(4/k) + o(1).$$

Set $k' = \sqrt{1-k^2}$. Then

$$K'(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k'^2\sin^2\theta}}\,d\theta.$$

We can separate the integral into two pieces.

$$K'(k) = \int_0^{\pi/2} \frac{k'\sin\theta}{\sqrt{1-k'^2\sin^2\theta}}\,d\theta + \int_0^{\pi/2} \frac{\sqrt{1-k'\sin\theta}}{\sqrt{1+k'\sin\theta}}\,d\theta$$

Set $\cos\theta = u$ to see that

$$\int_0^{\pi/2} \frac{k'\sin\theta}{\sqrt{1-k'^2\sin^2\theta}}\,d\theta = \int_0^1 \frac{k'}{\sqrt{k^2+k'^2u^2}}\,du$$
$$= \operatorname{arcsinh}\left(\frac{k'}{k}\right) = \ln\frac{k'+1}{k}.$$

(The last equality comes from the fact that $\sinh a = x$ is quadratic in e^a. It is easily shown that $e^a = x + \sqrt{x^2+1}$, hence $a = \operatorname{arcsinh} x = \ln(x+\sqrt{x^2+1})$.)

The second integral at $k = 0$ is

$$\int_0^{\pi/2} \frac{1-\sin\theta}{\cos\theta}\,d\theta.$$

This is $\ln 1 + \sin\theta\Big]_{\theta=0}^{\pi/2} = \ln 2$.

Consequently,

$$K'(k) \approx \ln\frac{k'+1}{k} + \ln 2 \approx \ln\frac{4}{k}.$$

5.4. **The series for K'.**

Define

$$\widetilde{K}(x) = K(\sqrt{x}) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\,\middle|\, x\right).$$

It is proved in Appendix 4 that

$$\Big((\delta+1)\partial - (\delta+1/2)(\delta+1/2)\Big)\widetilde{K}(x) = 0.$$

$$(x-x^2)\frac{d^2\widetilde{K}(x)}{dx^2} + (1-2x)\frac{d\widetilde{K}(x)}{dx} - \frac{1}{4}\widetilde{K}(x) = 0$$

Set $x \to 1-x$ to see that $\widetilde{K}(1-x) = K'(\sqrt{x})$ is also a solution.

Earlier two solutions of a hypergeometric differential equation were written out explicitly as

$$x^{1-c}{}_2F_1\left(\begin{matrix}a+1-c, b+1-c\\ 2-c\end{matrix}\,\middle|\, x\right) \qquad {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,\middle|\, x\right).$$

Unfortunately, when $c = 1$ these are the same solution.

We can use a clever and generally applicable trick. We have two solutions S_1 and S_2 which are distinct everywhere except at $c = c_0$. The idea is to differentiate on the parameter c. It works out as follows. Let z be $1-c$.

$$S_1 = x^z \frac{\Gamma(1/2+z)^2}{\Gamma(1+z)} {}_2F_1\left(\begin{matrix} \frac{1}{2}+z, \frac{1}{2}+z \\ 1+z \end{matrix} \middle| x \right) \qquad S_2 = \frac{\Gamma(1/2)^2}{\Gamma(1-z)} {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1-z \end{matrix} \middle| x \right).$$

$$S_1 = \sum_{m=0}^{\infty} \frac{\Gamma(1/2+m+z)\Gamma(1/2+m+z)}{\Gamma(1+m+z)\Gamma(1+m)} x^{m+z}$$

$$S_2 = \sum_{m=0}^{\infty} \frac{\Gamma(1/2+m)\Gamma(1/2+m)}{\Gamma(1+m)\Gamma(1+m-z)} x^m.$$

$$\frac{1}{2\pi}\frac{\partial}{\partial z}(S_2 - S_1) = \frac{1}{2}\sum_{m=0}^{\infty} \left(2\psi(1) - 2\psi(\tfrac{1}{2}) - \ln x \right) \frac{(\frac{1}{2})_m (\frac{1}{2})_m}{(1)_m m!} x^m$$

We have remembered again that $\Gamma(\frac{1}{2})^2 = \pi$.

In section 4 it was proved that

$$\psi(1) - \psi(\tfrac{1}{2}) = \ln 4.$$

So,

$$S(x) = \frac{1}{2\pi}\frac{\partial}{\partial z}(S_2 - S_1) = \ln(4) - \ln(x)/2 + o(1)$$

We have already proved that

$$K(\sqrt{1-k^2}) = \ln(4) - \ln(k) + o(1).$$

Then $\widetilde{K}(1-x) = K(\sqrt{1-x}) = \ln(4) - \ln(x)/2$ must be equal to $S(x)$.

This gives us the desired formula,

$$K'(k) = \frac{1}{2}\sum_{m=0}^{\infty} (2\psi(1) - 2\psi(\tfrac{1}{2}) - 2\ln k) \frac{(\frac{1}{2})_m (\frac{1}{2})_m}{(1)_m m!} k^{2m}$$

$$= -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+m+z)\Gamma(\frac{1}{2}+m+z)}{\Gamma(1+m+2z)\Gamma(1+m)} k^{2m+2z} \right]$$

6. Appell's F_4.

Appell generalized the hypergeometric function to two variables by considering the product of two ${}_2F_1$s on different variables,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n,$$

and combining pairs of parameters $(a)_m(a')_n$ into $(a)_{m+n}$, for example. We are concerned with the function F_4, in which the two upper pairs are combined.

$$F_4(a,b;c,c';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n\, m!\, n!}x^m y^n.$$

Suppose that the parameters are related by $a+b+1=c+c'$. We will prove a nice identity. The following argument is taken from (Slater 1964).

Let U_{mn} denote the coefficient of $x^m y^n$ in the function

$$\Phi(x,y)=(1-x)^{-a}(1-y)^{-b}F_4\left(a,b;c,a+b-c+1,\frac{x}{(1-x)(y-1)},\frac{y}{(1-x)(y-1)}\right).$$

We compute

$$\begin{aligned}[x^m y^n]x^r y^s(1-x)^{-a-r-s}(1-y)^{-b-r-s} &= [x^{m-r}y^{n-s}](1-x)^{-a-r-s}(1-y)^{-b-r-s}\\ &= (-1)^{r+s}\frac{(a+r+s)_{m-r}(b+r+s)_{m-s}}{(m-r)!(n-s)!}\end{aligned}$$

$$\begin{aligned}U_{mn} &= \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\frac{(a)_{r+s}\,(b)_{r+s}\,(a+r+s)_{m-r}\,(b+r+s)_{n-s}}{r!\,s!\,(c)_r\,(1+a+b-c)_s\,(m-r)!\,(n-s)!}\\ &= \frac{1}{m!\,n!}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(a)_{m+s}\,(-n)_s\,(b)_{n+r}\,(-m)_r}{r!\,(1+a+b-c)_s\,s!\,(c)_r}\\ &= \frac{(a)_m\,(b)_n}{m!\,n!}\,{}_2F_1\left(\left.\begin{matrix}a+m,-n\\1+a+b-c\end{matrix}\right|1\right){}_2F_1\left(\left.\begin{matrix}b+n,-m\\c\end{matrix}\right|1\right) \qquad \text{(A.1)}\end{aligned}$$

We proved Vandermonde's theorem for the case where one of the upper parameters is a negative integer.

$${}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\qquad {}_2F_1\left(\left.\begin{matrix}a,-n\\c\end{matrix}\right|1\right)=\frac{(c-a)_n}{(c)_n}.$$

Therefore,

$$\begin{aligned}\text{(A.1)} &= \frac{(a)_m\,(b)_n\,(1+b-c-m)_n\,(c-b-n)_m}{m!\,n!\,(1+a+b-c)_n\,(c)_m}\\ &= \frac{(a)_m\,(b)_n\,(1+b-c)_n\,(c-b)_m}{m!\,n!\,(c)_m\,(1+a+b-c)_m},\end{aligned}$$

where in the last line we have put $(c-b-n)_m=(c-b)_{m-n}(-1)^n(1-b+c)_n$ and used the identity $(1+b-c-m)_n=(-1)^n(c-b+m-n)_n$.

(In order for $(c-b)_{m-n}$ to make sense for all m, n, we very briefly have to require that $c-b$ is not an integer and then extend again by continuity.)

We have just shown that

$$U_{mn}=\frac{(a)_m\,(b)_n\,(1+b-c)_n\,(c-b)_m}{m!\,n!\,(c)_m\,(1+a+b-c)_n}.$$

Then

$$\Phi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m\,(c-b)_m}{m!(c)_m}\,\frac{(b)_n(1+b-c)_n}{n!(1+a+b-c)_n} x^m y^n.$$
$$= {}_2F_1(a, c-b; c; x)\,{}_2F_1(b, 1+b-c; 1+a+b-c; y)$$

Therefore, these two power series are identical. Apply Kummer's transform to both these ${}_2F_1$-s. The terms $(1-x)^{-a}$ and $(1-y)^{-b}$ are absorbed and

$$F_4\left(a, b; c, a+b-c+1; \frac{x}{(1-x)(y-1)}, \frac{y}{(1-x)(y-1)}\right)$$
$$= {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| \frac{x}{x-1}\right) {}_2F_1\left(\begin{matrix}a, b\\ 1+a+b-c\end{matrix}\middle| \frac{y}{y-1}\right)$$

Note that both sides of this equation are absolutely convergent even when expanded as power series of x and y, if x and y are small enough. Sufficient conditions are that $|x| < \frac{1}{2}$, $|y| < \frac{1}{2}$, and the two nonparameter arguments of the F_4 are in the domain of convergence. When this is true, the two sides are really identical, not just formally identical.

(In our case all the parameters are positive, which makes this obvious. In general, a different F_4 will majorize this series.)

This identity can also be written as follows.

$$F_4\Big(a, b; c, a+b-c+1; u(1-v), v(1-u)\Big)$$
$$= {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| u\right) {}_2F_1\left(\begin{matrix}a, b\\ 1+a+b-c\end{matrix}\middle| v\right)$$

where $u = x/(x-1), v = y/(y-1)$.

The left-hand side converges when $\sqrt{|u(1-v)|} + \sqrt{|v(1-u)|} < 1$ (at least); we will now prove this.

We are going to claim that the series $F_4(a, b; c, c'; x, y)$ converges when the sum $\sqrt{x} + \sqrt{y} < 1$. Clearly,

$$\Gamma(x) = \lim_{m\to\infty} \frac{m!\, m^{x-1}}{(x)_m} \qquad \text{implies} \qquad \frac{(a)_m}{m!} \sim \frac{1}{\Gamma(a)} m^{a-1}$$

We can get away with ignoring small m, n because sums for particular values of m and n converge like a ${}_2F_1$. For large m, n,

$$\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!} \sim \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)}(m+n)^{a+b} m^{-c} n^{-d} \binom{m+n}{m, n}^2$$

However,

$$\sum_{m+n \text{ constant}} \binom{m+n}{m}^2 |x|^m |y|^n \le (\sqrt{x} + \sqrt{y})^{2(m+n)}$$

because

$$\binom{2(m+n)}{2m, 2n} \ge \binom{m+n}{m, n}^2 \Leftrightarrow \binom{2(m+n)}{m+n} \ge \binom{2m}{m}\binom{2n}{n}$$

which can be proved by induction on n. Therefore, the F_4 converges better than

$$\sum_{s\ge 0} s^{|a|+|b|+|c|+|d|}(\sqrt{x} + \sqrt{y})^{2s} < \infty.$$

7. Modular equations

This section is taken from (Borwein 1987), and the subsections 7.1 and 7.2 follow very closely Section 4.1 in that book. The notation is similar but not identical.

This is a short introduction to the classical theory of modular equations.

We start with the following integral for K'. It was proved in section 5.2.

$$K'(k) = K(\sqrt{1-k^2}) = \int_0^\infty \frac{1}{\sqrt{1+x^2}\sqrt{k^2+x^2}}\,dx,$$

Write this as an integral over u, where $u = \frac{1}{2}(x - k/x)$. This is easier than it looks, since

$$(1+x^2)(k^2+x^2) = 4x^2\big(u^2 + \tfrac{1}{4}(1+k)^2\big).$$

We solve the quadratic equation and get $x = u + \sqrt{u^2+k}$. From this expression it is clear that x is monotonically increasing in u (and vice versa).

The derivative is

$$\frac{dx}{du} = \frac{x}{\sqrt{u^2+k}}.$$

So,

$$\int_0^\infty \frac{1}{\sqrt{1+x^2}\sqrt{k^2+x^2}}\,dx = \int_{-\infty}^\infty \frac{1}{2\sqrt{u^2+\frac{1}{4}(1+k)^2}\sqrt{u^2+k}}\,du.$$

The integral is symmetric around zero, so the integral over $[0,\infty]$ is just half the integral over $[-\infty,\infty]$. This absorbs the factor of $\frac{1}{2}$.

Substitute $u = \frac{1}{2}(1+k)z$.

$$\int_0^\infty \frac{1}{\sqrt{u^2+\frac{1}{4}(1+k)^2}\sqrt{u^2+k}}\,du = \frac{2}{1+k}\int_0^\infty \frac{1}{\sqrt{z^2+1}\sqrt{z^2+l^2}}\,dz$$

We have got an integral of the same form that we started with, but now the parameter is $l = 2\sqrt{k}/(1+k)$. By the arithmetic-geometric inequality, $l < 1$, and

$$K'(k) = \frac{2}{1+k}\int_0^\infty \frac{1}{\sqrt{z^2+1}\sqrt{z^2+l^2}}\,dz = \frac{2}{1+k}K'\left(\frac{2\sqrt{k}}{1+k}\right)$$

Set $k \to \dfrac{1-k'}{1+k'}$.

$$K'\left(\frac{1-k'}{1+k'}\right) = (1+k')K'(k).$$

Two more identities can be obtained by setting $k \to \sqrt{1-k^2}$. We make a table.

$$K'(k) = \frac{2}{1+k}K'\left(\frac{2\sqrt{k}}{1+k}\right) \qquad K(k) = \frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right)$$

$$K'(k) = \frac{1}{1+k'}K'\left(\frac{1-k'}{1+k'}\right) \qquad K(k) = \frac{1}{1+k}K\left(\frac{2\sqrt{k}}{1+k}\right)$$

Define $\tau(k) = \dfrac{K'}{K}(k)$. Then we can combine the upper-left and lower-right equations to get

$$\tau(k) = 2\tau\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{2}\tau\left(\frac{1-k'}{1+k'}\right)$$

It is evident that, since K' is monotonically decreasing and K is monotonically increasing, τ is also monotonically decreasing (and continuous). In the limit we

Table 1. Singular moduli

τ	k	
1	$1/\sqrt{2}$	0.7071
$\sqrt{2}$		0.4142
$\sqrt{3}$		0.2588
$\sqrt{4}$		0.1715
$\sqrt{5}$		0.1188
$\sqrt{6}$		0.0851
$\sqrt{7}$		0.0626
$\sqrt{8}$		0.0470

have $\tau(0) = \infty$ and $\tau(1) = 0$. Given $\tau \geq 0$, then, there exists exactly one value of k for which it is true that $\tau = \tau(k)$.

Lemma. If $\tau(k) = 1$, then $k = 1/\sqrt{2}$.
Obvious since $\sqrt{1-k^2} = k$, so $K(k) = K'(k)$. □

Lemma. If $\tau(k) = \sqrt{2}$, then $k = \sqrt{2} - 1$.
Because

$$\tau(k') = \frac{K}{K'}(x) = 1/\tau(k),$$

this is the same as finding a solution to the equation

$$\tau(k) = 2\tau(k'),$$

but this is true iff

$$k' = \frac{2\sqrt{k}}{1+k}.$$

This leads to a quartic,

$$1 - 2k - 2k^3 - k^4 = 0$$

or

$$(k^2+1)(k^2+2k-1) = 0.$$

Hence, $k = \sqrt{2} - 1$. □

7.1. **Classical modular equations.**
Pick an odd number n. Begin with the integrand in the definition of K,

$$\frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx.$$

Look for a polynomial P of degree $(n-1)/2$ so that the equation

$$\frac{1-y}{1+y} = \frac{(P(-x))^2(1-x)}{(P(+x))^2(1+x)}.$$

is invariant under the substitution $x \to 1/kx$, $y \to 1/ly$.
Then we will prove that

$$\frac{M(k,l)}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx = \frac{1}{\sqrt{1-y^2}\sqrt{1-l^2y^2}}\,dy,$$

where $M(k,l)$ depends only on k and l.

Solve for y in the equation above. It becomes

$$y = \frac{(P(+x))^2(1+x) - (P(-x))^2(1-x)}{(P(-x))^2(1-x) + (P(+x))^2(1+x)} = \frac{xU}{V}, \tag{7.1}$$

where U and V are polynomials in x^2, defined by the equations

$$V + xU = P(+x)^2(1+x).$$
$$V - xU = P(-x)^2(1-x).$$

That is to say, V is the even part, and xU is the odd part.

In order for the equation (7.1) to be invariant under the substitution $x \to 1/kx$, $y \to 1/ly$, it should be true that

$$\frac{1}{ly} = \frac{1}{kx}\frac{U(1/kx)}{V(1/kx)} \qquad\qquad y = \frac{kx}{l}\frac{V(1/kx)}{U(1/kx)}.$$

Let $cf(x)$, the *complement* of f, denote $(kx)^{n-1}f(1/kx)$ for any function f. If f is a polynomial of degree $n-1$, $f(x) = p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_1 x + p_0$, then

$$cf = p_{n-1} + p_{n-2}(kx) + \cdots + p_1(kx)^{n-2} + p_0(kx)^{n-1}.$$

In particular, $ccf = k^{n-1}f$. Define $\hat{c}f = k^{-(n-1)/2}cf$, so that the outcome of $\hat{c}\hat{c}f$ is just the function f again. (This is true for any function f, since $ccf = c[(kx)^{n-1}f(1/kx)] = (kx)^{n-1}/x^{n-1}f(x)$.)

We ask for the following to be true.

$$\frac{U}{V} = \frac{y}{x} = \frac{k}{l}\frac{\hat{c}V}{\hat{c}U}.$$

This will be true if

$$U = \sqrt{\frac{k}{l}}\hat{c}V.$$

This is a set of $(n-1)/2$ quadratic equations in the coefficients of P. (Both sides are even polynomials of degree $n-1$.) We can solve these equations parametrically for degree 3. See section 7.2.

Assume that P, k, l have been chosen so that (7.1) has the property stated above. Then

$$1 - y = \frac{P(-x)^2}{V}(1-x) \qquad\qquad 1 + y = \frac{P(+x)^2}{V}(1+x)$$
$$1 - ly = \sqrt{\frac{l}{k}}\frac{\hat{c}[P(-x)^2]}{V}(1-kx) \qquad\qquad 1 + ly = \sqrt{\frac{l}{k}}\frac{\hat{c}[P(+x)^2]}{V}(1+kx)$$

The first row follows from $y = xU/V$. The second row follows from the computation

$$V - lxU = \hat{c}\left(\hat{c}V - \frac{l}{kx}\hat{c}U\right) = \hat{c}\sqrt{\frac{l}{k}}\left(U - \frac{1}{x}V\right) = \sqrt{\frac{l}{k}}\hat{c}[P(-x)^2](1-kx)$$

We define the polynomial Q by

$$Q(x)^2 = \sqrt{\frac{l}{k}}\hat{c}[P(x)^2] \qquad \text{or} \qquad Q(x) = \sqrt[4]{\frac{l}{k}}k^{(n-1)/4}x^{(n-1)/2}P\left(\frac{1}{kx}\right).$$

Then

$$1-y=\frac{P(-x)^2}{V}(1-x) \qquad\qquad 1+y=\frac{P(+x)^2}{V}(1+x)$$
$$1-ly=\frac{Q(-x)^2}{V}(1-kx) \qquad\qquad 1+ly=\frac{Q(+x)^2}{V}(1+kx)$$

It is clear that $y=0$ when $x=0$ and $y=1$ when $x=1$. We now make the assumption that the derivative dy/dx is positive when $0<x<1$. We have to check this for the particular U, V that appear in section 7.2.

This makes $x \mapsto y : [0,1] \to [0,1]$ an increasing bijection. Therefore,

$$\int_0^1 \frac{1}{\sqrt{1-y^2}\sqrt{1-l^2y^2}}\,dy = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,\frac{(xU)'V-xUV'}{P^-\,P^+\,Q^-\,Q^+}\,dx.$$

We have written e.g. P^- for $P(-x)$.

All we need to do now is prove that this second factor, which we will call M,

$$M=\frac{(xU)'V-xUV'}{P^-\,P^+\,Q^-\,Q^+},$$

does not depend on x.

Note that the polynomials $V-lxU=V(1-ly)$ and $V+lxU=V(1+ly)$ are divisible by Q^- and Q^+ respectively. Since

$$\begin{aligned}(xU)'V-xUV' &= \frac{1}{2}(V+xU)'(V-xU)-\frac{1}{2}(V-xU)'(V+xU)\\ &= \frac{1}{2l}(V+lxU)'(V-lxU)-\frac{1}{2l}(V-lxU)'(V+lxU)\end{aligned}$$

the numerator of M is divisible by both $P^-\,P^+$ and $Q^-\,Q^+$.

If they are not coprime, then $V^2-x^2U^2$ and $V^2-l^2x^2U^2$ share a common factor, so V and xU are not coprime.

Assume that V and xU are coprime (again, we have to check this later). Then $P^-\,P^+$ and $Q^-\,Q^+$ are coprime, and the numerator of M is divisible by the product $P^-\,P^+\,Q^-\,Q^+$. Hence M is a polynomial, and by comparing the degrees of the numerator and denominator, it must be constant.

Therefore,

$$M \equiv \frac{U(0)\,V(0)}{P(0)^2\,Q(0)^2}=\frac{U(0)}{V(0)}$$

since $P(0)^2Q(0)^2=(V-xU)(V-lxU)/(1-x)(1-lx)\big]_{x=0}=V(0)^2$.

Theorem. Fix a degree n. Suppose that the polynomials U and V, defined as above, solve the equation $U=(k/l)^{1/2}\hat{c}V$. If $y=xU/V$ is increasing on $[0,1]$, and xU and V have no common factor (for fixed k, l), then

$$\int_0^1 \frac{1}{\sqrt{1-y^2}\sqrt{1-l^2y^2}}\,dy = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,\frac{U(0)}{V(0)}\,dx.$$

In the language of elliptic integrals,

$$K(l)=\frac{U(0)}{V(0)}K(k).$$

Note that the equation relating U to V is only solvable for certain k and l. For $n=3$, once k is chosen, the value of l is uniquely determined. See the next page.

□

7.2. **The cubic modular equation.**

Suppose that the degree of P is one, so that $P = a+bx$. Scaling P by a constant makes no difference, so we will write $P = 1 + \alpha x$. Then

$$\begin{aligned} V + xU &= 1 + (2\alpha+1)x + (\alpha^2+2\alpha)x^2 + \alpha x^3. \\ V &= 1 + (\alpha^2+2\alpha)x^2 \\ U &= (2\alpha+1) + \alpha^2 x^2 \end{aligned}$$

Writing out $U = (k/l)^{1/2}\hat{c}V$, we get the two equations

$$\alpha^2 = \sqrt{\frac{k^3}{l}} \qquad\qquad (2\alpha+1) = \frac{1}{\sqrt{lk}}(\alpha^2+2\alpha)$$

or

$$\frac{\alpha(\alpha+2)}{(2\alpha+1)} = \sqrt{lk}.$$

Hence

$$\sqrt{\frac{\alpha^3(\alpha+2)}{(2\alpha+1)}} = k \qquad\qquad \sqrt{\frac{\alpha(\alpha+2)^3}{(2\alpha+1)^3}} = l$$

and

$$\sqrt{kl} + \sqrt{k'l'} = \frac{\alpha(\alpha+2)}{(2\alpha+1)} + \left(\frac{(1-\alpha^2)(1+\alpha)^2}{(2\alpha+1)}\frac{(1-\alpha^2)(1-\alpha)^2}{(2\alpha+1)^3}\right)^{1/4} = 1.$$

Here α is a parameter, and we should check that α is uniquely defined in terms of k when $0 \le k \le 1$.

To derive the above equation, we compute

$$l' = \sqrt{1-l^2} = \sqrt{\frac{(1-\alpha)^2(1-\alpha^2)}{(2\alpha+1)^3}}$$

and a similar equation for k'. Since l' and k' satisfy the relationship above, we suspect that they are also related parametrically, and we look for β which will give $k(\beta) = l'$ and $l(\beta) = k'$ when substituted into our parametrization.

Let β be the root of

$$(2\alpha+1)(2\beta+1) = 3 \qquad \text{or} \qquad \alpha = \frac{1-\beta}{2\beta+1}.$$

Then

$$l' = \sqrt{\frac{\beta^3(\beta+2)}{2\beta+1}} = k(\beta), \qquad k' = \sqrt{\frac{\beta(\beta+2)^3}{(2\beta+1)^3}} = l(\beta).$$

So l' and k' are related, but with a different parameter, β.

In section 7.1 it was proved that

$$K(l) = \frac{U(0)}{V(0)}K(k) = (2\alpha+1)K(k)$$

under two assumptions. First,

$$\begin{aligned}(xU)'V &= ((2\alpha+1)+3\alpha^2x^2)(1+(\alpha^2+2\alpha)x^2)\\ xUV' &= 2x^2((2\alpha+1)+\alpha^2x^2)(\alpha^2+2\alpha)\\ (xU)'V - xUV' &= \alpha^3(\alpha+2)x^4 + [-\alpha(2\alpha+1)(\alpha+2)+3\alpha^2]x^2 + (2\alpha+1)\\ &= \alpha^3(\alpha+2)x^4 - (2\alpha^3+2\alpha^2+2\alpha)x^2 + (2\alpha+1)\\ &= (\alpha(\alpha+2)x^2 - (2\alpha+1))(\alpha^2x^2-1)\end{aligned}$$

But $(2\alpha+1)/\alpha(\alpha+2) > 1$ when $0 < \alpha < 1$, so there are no roots of this polynomial between zero and one.

We could have got this result by considering P and Q, since it is not necessary to make the first assumption to prove that M is a constant. You can check that

$$(xU)'V - xUV' = (2\alpha+1)P^-\,P^+\,Q^-\,Q^+$$

if you like.

The inequality $(2\alpha+1)/\alpha(\alpha+2) > 1$ also shows that V and xU are coprime, which was the second assumption.

We get to use the result of section 7.1 to produce the equations

$$K'(k) = (2\beta+1)K'(l) \qquad K(k) = (2\alpha+1)^{-1}K(l),$$

and

$$\frac{K'}{K}(k) = (2\beta+1)(2\alpha+1)\frac{K'}{K}(l),$$

but since we know that $(2\beta+1)(2\alpha+1) = 3$,

$$\tau(k) = 3\tau(l).$$

The relation between the moduli k and l is parametric, but above we showed that they are also related by

$$\sqrt{kl} + \sqrt{k'l'} = 1.$$

Let $0 < k < 1$. The derivative of the expression $F(x) = \sqrt{kx}+\sqrt{k'x'}$ is monotonic in x. It is clear that $F(0)$ and $F(1) \leq 1$, and $F(k) = k+k' \geq 1$. Therefore, $F(x) = 1$ has precisely two roots.

One of the roots is l. To find the other root, consider the equation $3\tau(m') = \tau(k')$, which has exactly one solution $m \in (0,1)$. We have already proved that m' and k' are related by

$$\sqrt{m'k'} + \sqrt{mk} = 1,$$

so m is the other root.

Since $3\tau(m') = \tau(k')$, it follows that $\tau(m) = 3\tau(k)$. Hence $m < k < l$.

7.3. **The roots of $\sqrt{kl} + \sqrt{k'l'} = 1$.** Given $k \in (0,1)$, let $l_1 > k$ and $l_2 < k$ be the two roots of

$$\sqrt{kl} + \sqrt{k'l'} = 1$$

in the interval $(0,1)$. Then

$$\tau(k) = 3\tau(l_1) = \frac{1}{3}\tau(l_2).$$

7.4. **The sixth singular modulus.**

The *sixth singular modulus* is the value of k which satisfies the equation

$$\tau(k) = \frac{K'}{K}(k) = \sqrt{6}.$$

Let it be called k_6, and also let $k_{2/3}$ be the solution of $\tau(k) = \sqrt{2/3}$. Since $\tau(k')$ is the reciprocal of $\tau(k)$, the equation above can also be written $\tau(k) = 6\tau(k')$.

We proved in section 7.1-7.2 that, when k and l are in the interval $(0, 1)$ and

$$\sqrt{kl} + \sqrt{k'l'} = 1$$

then $\tau(k) = 3\tau(l)$ or $\tau(k) = \frac{1}{3}\tau(l)$. This allows us to deduce the value of k_6.

The following argument is copied from (Berndt and Chan). It is a special case of a more general argument due to Ramanujan. Substitute

$$k \to \frac{2\sqrt{x}}{1+x} \qquad l \to x'.$$

The above equation becomes

$$\sqrt[4]{\frac{4x(1-x)}{1+x}} + \sqrt{\frac{x(1-x)}{1+x}} = 1. \tag{7.2}$$

When x is a root of this equation then $\tau(k) = 3^{\pm 1}\tau(l)$ becomes

$$\tau\left(\frac{2\sqrt{x}}{1+x}\right) = 3^{\pm 1}\tau(x'),$$

or, referring to the transformation formulae at the beginning of section 7,

$$\tau(x)^2 = 2 \times 3^{\pm 1}.$$

Therefore, if x is a root of (7.2), then $\tau(x) = \sqrt{6}$ or $\tau(x) = \sqrt{2/3}$. Set

$$u = \sqrt[4]{\frac{4x(1-x)}{1+x}}.$$

Then (7.2) becomes

$$u + \frac{1}{2}u^2 = 1,$$

which is readily solved. We find that the positive root is $u = \sqrt{3} - 1$.

Solve

$$\frac{4x(1-x)}{1+x} = u^4 = 28 - 16\sqrt{3} = 4(2-\sqrt{3})^2.$$

This is conveniently written as

$$x^2 + (c-1)x + c = 0,$$

where $c = (2-\sqrt{3})^2$ is defined in section 3. The solution of this equation is

$$x = \frac{1-c}{2} \pm \frac{\sqrt{(c-1)^2 - 4c}}{2} = \frac{1-c}{2} \pm \sqrt{2c} = 2\sqrt{3} - 3 \pm (2\sqrt{2} - \sqrt{6}).$$

In fact,

$$k_6 = 2\sqrt{3} - 3 - 2\sqrt{2} + \sqrt{6} = (2-\sqrt{3})(\sqrt{3}-\sqrt{2}) = \sqrt{v}$$
$$k_{2/3} = 2\sqrt{3} - 3 + 2\sqrt{2} - \sqrt{6} = (2-\sqrt{3})(\sqrt{3}+\sqrt{2}) = \sqrt{1-u}$$

in the notation of section 3.

We now plug these back into the argument.

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